Similarity and Trigonometry

**Unit Overview**
In this unit, you will study special right triangles and right triangle trigonometry. You will also study similarity transformations and similarity in polygons.

**Key Terms**
As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

**Academic Vocabulary**
- triangulation

**Math Terms**
- dilation
- center of dilation
- similarity transformation
- similar
- indirect measurement
- Triangle Proportionality Theorem
- Parallel Proportionality Theorem
- Right Angle Altitude Theorem
- geometric mean
- Pythagorean Theorem
- Pythagorean triple
- opposite leg
- adjacent leg
- trigonometric ratio
- sine
- cosine
- tangent
- inverse trigonometric function
- Law of Sines
- Law of Cosines

**ESSENTIAL QUESTIONS**
How are similar triangles used in solving problems in everyday life?
What mathematical tools do I have to solve right triangles?

**EMBEDDED ASSESSMENTS**
These embedded assessments, following Activities 18, 21, and 23, allow you to demonstrate your understanding of similarity, proportionality, special right triangles, and right angle trigonometry.

**Embedded Assessment 1:**
Similarity in Polygons p. 273

**Embedded Assessment 2:**
Right Triangles p. 301

**Embedded Assessment 3:**
Trigonometry p. 331
Write your answers on notebook paper. Show your work.

1. Draw a graph to show how the figure is transformed by the function. Then classify the transformation as rigid or nonrigid.
   a. \((x, y) \rightarrow (x - 2, y + 1)\)
   
   [Graph showing transformation]

   b. \((x, y) \rightarrow (0.5x, 0.5y)\)
   
   [Graph showing transformation]

2. Simplify.
   a. \(\sqrt{72}\)
   b. \(\frac{7}{\sqrt{5}}\)

3. Solve the following for \(x\).
   a. \(5 = \frac{2}{x}\)
   b. \(x^2 + 3x + 2 = 0\)
   c. \(\frac{\sqrt{2}}{\sqrt{7}} = \frac{x}{\sqrt{14}}\)

4. Find the distance between \((3, 5)\) and \((7, -1)\).

5. Find the lengths \(AB\) and \(BD\) if \(AD = 29\) units.

   \[A \quad 2x + 4 \quad B \quad 2x \quad C \quad x \quad D\]

6. Solve the following equation for \(a\).
   \[\frac{2a + b}{c} = d\]

7. Evaluate \(4\sqrt{2} + 2(\sqrt{36} - \sqrt{8})\).

8. Find the length of side \(a\) in the right triangle pictured.

   [Right triangle diagram]

9. In the figure below, \(c\) is a transversal cutting parallel lines \(l\) and \(m\). List at least four geometric relationships that exist among angles 4, 5, and/or 8.
Dilations and Similarity Transformations
Scaling Up/Scaling Down
Lesson 17-1 Dilations

Learning Targets:
• Perform dilations on and off the coordinate plane.
• Describe dilations.

SUGGESTED LEARNING STRATEGIES: Create Representations, Visualization, Summarizing

Graphic artists create visual information for electronic and print media. They use graphic design software in their daily work. For example, a graphic artist can use computer software to design a brochure. The text and art for the pages of the brochure can be moved around or scaled up and down before the actual brochure is printed.

Kate is a graphic artist. She wants to create two different layouts of a brochure for her client to review.

The art for the layouts is shown.

1. What are the coordinates of the vertices of the pre-image shown in Layout A?

2. What are the coordinates of the vertices of the image shown in Layout B?

3. What relationship do you notice between the coordinates for Layout A and the coordinates for Layout B?

4. Express the transformation in Item 3 as a function.
5. Kate created a third layout for the client to review. What are the coordinates of the vertices of the image shown in Layout C?

![Diagram of Layout C]

6. What relationship do you notice between the coordinates for Layout A and the coordinates for Layout C?

7. Express the relationship in Item 6 as a function.

A **dilation** is a transformation that changes the size of a figure, but not its shape, by a scale factor $k$. The scale factor determines whether the transformation is a reduction or an enlargement.

8. a. What is the scale factor for the transformation of Layout A to Layout B?

b. Is Layout B a reduction or an enlargement of Layout A?

c. What is the scale factor for the transformation of Layout A to Layout C?

d. Is Layout C a reduction or an enlargement of Layout A?
Lesson 17-1
Dilations

9. In general, for what values of scale factor $k$ is a dilation a reduction?

10. In general, for what values of scale factor $k$ is a dilation an enlargement?

11. Express regularity in repeated reasoning. A pre-image point, $P$, has coordinates $(a, b)$.
   a. How can you find the coordinates of the image of $P$ after a dilation with scale factor $k$?

   b. Express the relationship in item a as a function.

Recall the composition of transformations is a series of two or more transformations performed on a figure one after another.

12. Rectangle $A(0, 6), B(4, 6), C(4, 0), D(0, 0)$ is mapped onto rectangle $A'B'C'D'$ by a composition of transformations. Identify the coordinates of rectangle $A'B'C'D'$ after the translation $(x, y) \rightarrow (x - 4, y - 6)$ followed by a dilation with scale factor $\frac{1}{2}$.

Check Your Understanding

13. Triangle $R(0, 0), S(0, 4), T(3, 0)$ is mapped onto $\triangle R'S'T'$ by a dilation.
   a. If $\frac{R'S'}{RS} = \frac{5}{2}$, is $\triangle R'S'T'$ a reduction or an enlargement? Explain.

   b. If $\frac{R'S'}{RS} = \frac{5}{2}$ and the coordinates of $\angle R'$ are $(0, 0)$, what are the coordinates of $\angle S'$?

   c. Triangle $RST$ is mapped onto $\triangle R'S'T'$ by a composition of transformations. Triangle $RST$ is dilated by the scale factor $\frac{5}{2}$ followed by a translation $(x, y) \rightarrow (x + 2, y - 5)$. What are the coordinates of $\triangle R'S'T'$?

14. Describe a dilation with a scale factor of 1.

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In each of Kate's layouts above, the center of dilation was the origin. Now she considers some dilations where the center is not the origin.

Kate wants to dilate triangle $ABC$ with a scale factor of 2 and center of dilation $(-2, 4)$.

When the origin is not the center of a dilation, you need to subtract the coordinates of the center of dilation from the coordinates of the pre-image, multiply this difference by the scale factor, and then add the coordinates of the center of dilation.

If the point $(x, y)$ lies on the pre-image and the figure is dilated with a scale factor of $k$ with $(a, b)$ as the center of dilation, the corresponding point on the image is $(a + k(x - a), b + k(y - b))$.

15. What are the coordinates of the vertices of the pre-image?

16. What are the coordinates of the vertices of the image?

17. Graph the pre-image and image of the dilation below.
Lesson 17-1
Dilations

Check Your Understanding

18. Triangle T(6, 4), U(−4, 2), V(2, −3) is mapped onto △T′U′V′.
   a. Triangle TUV is mapped onto △T ′U ′V ′ by a dilation with a scale factor of \( \frac{1}{2} \) and center (2, −3). What are the coordinates of △T ′U ′V ′?
   b. Triangle TUV is mapped onto △T ′U ′V ′ by a dilation with a scale factor of \( \frac{1}{2} \) and center (2, −3) followed by a reflection over the x-axis. What are the coordinates of △T ′U ′V ′?

19. A point (x, y) is dilated by a scale factor of 4 with center (−2, 1). What are the coordinates of the image?

You can use geometry software to explore dilations.

20. To perform a dilation, first plot a point for the center of dilation.
   a. Draw triangle ABC and dilate it using a scale factor of 2.
   b. Compare and contrast the pre-image and the image.

21. Perform dilations on triangle ABC using different scale factors. Use \( k > 1 \), \( k = 1 \), and \( 0 < k < 1 \).
   a. Describe how, in general, a dilation transforms a figure.
   b. Do dilations preserve angle measures? Justify your answer using your drawing.
   c. Is a dilation a rigid transformation? Explain.

MATH TIP
A rigid transformation keeps the same distance between the points that are transformed; the shape and size of the pre-image and image are the same.
Dilations don’t have to be performed on the coordinate plane or with the use of software. Here is a more general definition of a dilation.

Given a point $O$ and a positive real number $k$, the dilation with center of dilation $O$ and scale factor $k$ maps point $P$ to $P'$, where $P'$ is the point on $OP$ such that $OP' = k \cdot OP$.

A dilation centered at point $O$ with scale factor $k$ can be written in function notation as $D_{O,k}(P) = P'$.

The figure shows a dilation with a scale factor of 3 because $OP' = 3 \cdot OP$.

22. How would you write the dilation above in function notation?

23. $\triangle XYZ$ is an isosceles triangle with a base angle of $65^\circ$. The triangle is dilated by a factor of $\frac{2}{3}$. What are the measures of the angles of $\triangle X'Y'Z'$? Explain your reasoning.

24. A dilation is centered at point $O$ with a scale factor of $\frac{1}{4}$, such that $OP' = \frac{1}{4} OP$. Write the dilation in function notation.
LESSON 17-1 PRACTICE

25. Given a triangle and its image under a dilation, explain how you can use a ruler to determine the scale factor.

26. A graphic artist has enlarged a rectangular photograph using a scale factor of 4. The perimeter of the enlargement is 144 in. What is the perimeter of the original photograph?

27. A photographer enlarged a picture. If the width of the image is 5 inches and the width of the pre-image was \(x\), what is the scale factor for the dilation in terms of \(x\)?

28. Sketch the pre-image with the given vertices, and sketch the image with the given scale factor and center of dilation at the origin.
   \(A(2,4), B(-3,-1), C(0,1)\); scale factor of 3

29. Sketch the pre-image with the given vertices, and sketch the image with the given scale factor and center of dilation at the origin.
   \(A(-2,3), B(2,3), C(2,-1), D(-2,-1)\); scale factor of \(\frac{1}{4}\)

30. Sketch the image of a line segment with endpoints \(A(-2,-5)\) and \(B(1,-1)\) under a dilation centered at the origin with a scale factor of 1.5.

31. Sketch the image of a line segment with endpoints \(A(0,0)\) and \(B(5,4)\) under a dilation centered at the origin with a scale factor of \(\frac{1}{2}\).

32. **Reason abstractly.** A dilation maps each point \((x, y)\) of the pre-image to \((-5 + \frac{1}{2}(x + 5), 8 + \frac{1}{2}(y - 8))\). What are the scale factor and center of dilation for the transformation?
Learning Targets:

- Understand the meaning of similarity transformations.
- Use similarity transformations to determine whether figures are similar.

**SUGGESTED LEARNING STRATEGIES:** Create Representations, Graphic Organizer, Predict and Confirm, Visualization, Think-Pair-Share

Another of Kate's clients has decided he wants the picture shown in Layout A enlarged and repositioned so that it looks like Layout B. Kate needs to transform the picture in Layout A to Layout B.


3. Describe in words a composition that would map Layout A to Layout B.

4. Describe the composition you used in Item 3 in function notation.
Lesson 17-2
Similarity Transformations

5. Perform the following compositions of transformations on the pre-image, \( \triangle ABC \). Draw the pre-image and image on the grids.

a. \( D_{O,2}(r_{y=1}) \)

b. \( R_{90^\circ}(D_{O,\frac{1}{4}}) \)

c. \( D_{O,4}(T_{(2,-2)}(D_{O,\frac{1}{2}})) \)
6. What can you conclude about the pre-image and image of each of the figures in Item 5?

A similarity transformation is a transformation which results in the pre-image and image having the same shape.

7. Model with mathematics. Complete the Venn diagram using each of the transformations on the right.

Two figures are similar if and only if one figure can be transformed from the other figure using one or more similarity transformations.

8. Kate wants to make sure the two figures below are similar. Use transformations to justify that the two figures are similar. Describe the transformations using words and function notation.
Lesson 17-2
Similarity Transformations

Check Your Understanding

9. What single dilation produces the same image as the composition of dilations \( D_{0.6} (D_{0.5} (D_{0.2.3})) \)?

10. Explain why rigid motions are similarity transformations.

LESSON 17-2 PRACTICE

11. Use geometry software or grid paper to perform the following compositions of transformations on the pre-image.
   a. \( D_{0.3} (r_{x=y}) \)
      pre-image:
      ![Pre-image for a]
   b. \( R_{0,90^\circ} (D_{0.1}) \)
      pre-image:
      ![Pre-image for b]

12. Make sense of problems. Determine whether the figures are similar. If they are, describe in words two sequences of similarity transformations that could be used to map the pre-image to the image.

   ![Image of figures for 12]
Learning Targets:
- Identify properties of similar figures.
- Apply properties of similar figures.

**SUGGESTED LEARNING STRATEGIES:** Create Representations, Predict and Confirm, Visualization, Create a Plan, Look for a Pattern

1. The two figures shown below are similar. The symbol for similarity is ~.
   So you can write the similarity statement $ABCD \sim A'B'C'D'$.
   Rectangle $ABCD$ has coordinates $(-1, 0)$, $(-1, 3)$, $(5, 3)$, and $(5, 0)$.

   ![Diagram of rectangles $ABCD$ and $A'B'C'D'$]

   **a.** Describe a sequence of similarity transformations that maps $ABCD$ to $A'B'C'D'$.

   **b.** Do you get the same image if you perform the similarity transformations in a different order?

2. Each similarity transformation described in Item 1 preserves angle measures, so corresponding angles of similar figures are ________.

3. Write the ratios of the corresponding side lengths of the pre-image and image. What do you notice about the ratios?
4. Kate is about to meet with her final client of the day. She needs to create a wall mural. She can measure the length of the wall but not the height. She has a prototype picture and knows that the wall and the picture are similar rectangles. Kate needs to find the area of the wall.

![Wall and Prototype Image]

**Wall**
- 14 ft

**Prototype**
- 18 in.
- 31.5 in.

a. Which sides of the two rectangles are corresponding?

b. Write two different proportions that can be used to find the height of the wall.

c. **Reason abstractly and quantitatively.** Can the proportion \( \frac{x}{18} = \frac{31.5}{168} \) be used to find the height of the wall? Explain why or why not.

d. What is the height of the wall? Show your work.

e. What is the area of the wall? Show your work.
Lesson 17-3
Properties of Similar Figures

Check Your Understanding

5. Rectangle \( \text{DEFG} \sim \text{rectangle } \text{WXYZ} \). Which side lengths must be proportional? Write the appropriate proportions.

6. If the two rectangles are similar, what is the value of \( x \)?

\[
\frac{12}{x} = \frac{18}{9}
\]

LESSON 17-3 PRACTICE

For each pair of similar figures, find the missing side length.

7. \[
\begin{array}{c}
24 \\
28 \\
y
\end{array}
\quad
\begin{array}{c}
16 \\
\quad
\end{array}
\]

8. \[
\begin{array}{c}
2.6 \\
4.8 \\
z
\end{array}
\quad
\begin{array}{c}
3.6
\end{array}
\]

9. \( \triangle ABC \sim \triangle DEF \), \( m\angle A = 45^\circ \), and \( m\angle C = 90^\circ \). Classify \( \triangle DEF \) as precisely as possible.

10. Explain how the scale factor in Item 1a relates to the similarity ratio in Item 3.

11. Critique the reasoning of others. Mario’s teacher wrote the statement \( \triangle PQR \sim \triangle STU \) on the board. She asked him to write a proportion that must be true. Mario wrote \( \frac{PQ}{QR} = \frac{ST}{TU} \). Do you agree with Mario’s proportion? Justify your answer.
ACTIVITY 17 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 17-1

1. Sketch the image of each figure with the given vertices and scale factor and center of dilation at the origin.
   a. \(A(0, 1), B(4.5, 0), C(2, 2)\); scale factor of 2
   b. \(B(10, 6), C(8, 5), D(-7, 9)\); scale factor of \(\frac{1}{2}\)
   c. \(B(-4, 6), C(6, 6), D(6, 3), E(-4, 3)\); scale factor of \(\frac{1}{5}\)

2. Sketch the image of a line segment with endpoints \(A(-1, 4)\) and \(B(2, 3)\) under a dilation centered at the origin with a scale factor of \(\frac{1}{2}\).

3. Sketch the image of a triangle with vertices \((-3, 2), (0, 0),\) and \((2, 1)\) under a dilation centered at \((2, 1)\) with a scale factor of 3.

4. What is the scale factor of the dilation centered at the origin that maps \(\triangle ABC\) to \(\triangle A'B'C'\)?

5. A graphic artist has reduced a rectangular photograph using a scale factor of \(\frac{1}{6}\). The perimeter of the original photograph is 165 in. What is the perimeter of the reduced photograph?

6. What are the coordinates of \(A\) after the figure undergoes a dilation with a scale factor of 1.2 centered at the origin?

Lesson 17-2

7. What single dilation produces the same image as the composition of dilations \(D_{O, \frac{2}{3}}(D_{O, \frac{3}{5}})(\triangle ABC)\)?
   A. \(D_{O, 12}\)
   B. \(D_{O, 6}\)
   C. \(D_{O, 2}\)
   D. \(D_{O, \frac{5}{3}}\)
8. Use geometry software to perform the following compositions of transformations on the pre-image.
   a. $D_{0,2}(r_{y = 0})$
      pre-image:

   b. $T_{(1,2)}(D_{0,1}(R_{0,90^\circ}))$
      pre-image:

9. Determine whether the figures are similar. If they are, write, in function notation, the sequence of similarity transformations that maps the pre-image to the image.
   a. 

   b. 

Lesson 17-3

10. Find the indicated measurement for each pair of similar figures.
   a. 

   b. 

MATHEMATICAL PRACTICES

Attend to Precision

11. The vertices of a quadrilateral have the coordinates $A(-3, 0), B(-8, -6), C(8, 4),$ and $D(2, 4).$ After a dilation with a scale factor of 5, the vertices are translated $T(1, -3).$ What are the coordinates of the vertices of the final image?
Learning Targets:
- Develop criteria for triangle similarity.
- Prove the AA similarity criterion.

LEARNING STRATEGIES: Close Reading, Marking the Text, Questioning the Text, Think-Pair-Share, Create Representations, Visualization

A land boundary dispute exists between a national park and a bordering landowner. Surveyors have been hired to determine land boundaries and help settle the dispute. The surveyors will use similar triangles to find measurements.

You learned in Unit 2 that it is not always necessary to use the definition of congruence to determine if triangles are congruent, because there are congruence criteria you used to show two triangles are congruent. In a similar way, you can look for criteria for showing triangles are similar.

1. Recall that two figures are similar if and only if their corresponding angles are congruent and their corresponding side lengths are proportional.

   a. Use appropriate tools strategically. Use a ruler and protractor to draw two noncongruent triangles, \( \triangle ABC \) and \( \triangle DEF \), with angle measures 45°, 60°, and 75°. Then complete the table.

   \[
   \begin{array}{ccc}
   \triangle ABC & \triangle DEF & \text{Ratios of Lengths of Corresponding Sides} \\
   AB = & DE = & \frac{AB}{DE} = \\
   BC = & EF = & \frac{BC}{EF} = \\
   CA = & FD = & \frac{CA}{FD} = \\
   \end{array}
   \]

   b. Do you think three pairs of congruent angles guarantee triangle similarity? Justify your response.

   c. Use a ruler and protractor to draw two noncongruent triangles, \( \triangle QRS \) and \( \triangle XYZ \), with angle measures of 45° and 60°. Then complete the table.

   \[
   \begin{array}{ccc}
   \triangle QRS & \triangle XYZ & \text{Ratios of Lengths of Corresponding Sides} \\
   QR = & XY = & \frac{QR}{XY} = \\
   RS = & YZ = & \frac{RS}{YZ} = \\
   SQ = & ZX = & \frac{SQ}{ZX} = \\
   \end{array}
   \]

   d. Look for and express regularity in repeated reasoning. Without measuring, could you have predicted that the triangles in Item 1c would be similar? Justify your response.
1. Do you think two pairs of congruent angles guarantee triangle similarity? Explain.

2. The investigation you did in Item 1 suggests the AA Similarity Postulate for triangles. State the postulate.

**Check Your Understanding**

3. Write whether AA similarity can be used to show that the pair of triangles are similar. Explain your answer.
   a. 
   b. 

You can use the AA Similarity Postulate to show that two triangles are similar. In order to prove why this criterion works, you must show that it follows from the definition of similarity in terms of rigid motions.

4. To justify the AA Similarity Postulate, consider the two triangles below.

   a. What given information is marked in the figure?
   b. Based on the definition of similarity, explain whether transformations can show that $\triangle ABC \sim \triangle DEF$.
   c. Draw a figure to show the result of a dilation that maps $\triangle ABC$ to $\triangle A'B'C'$ using the scale factor $k = \frac{DE}{AB}$.
**Lesson 18-1**

**Similarity Criteria**

**d.** By the definition of similarity transformations, complete the following statements.

\[ \angle A' \cong \angle B' \cong \angle A'B' = k \cdot AB = \square \]

**e.** By the Transitive Property of Congruence, \( \angle A' \cong \angle D \) and \( \square \).

**f.** How are \( \triangle A'B'C' \) and \( \triangle DEF \) related? How do you know?

The argument in Items 4d through 4f shows that a sequence of rigid motions maps \( \triangle A'B'C' \) to \( \triangle DEF \). The dilation in Item 4c followed by the sequence of rigid motions shows that there is a sequence of similarity transformations that maps \( \triangle ABC \) to \( \triangle DEF \). So, by the definition of similarity, \( \triangle ABC \sim \triangle DEF \).

You learned there were six criteria for showing that triangles are congruent, so it seems logical that there may be other criteria for showing that triangles are similar.

**5.** First investigate whether there is an SAS similarity criterion.

**a.** How do you think the SAS similarity criterion differs from the SAS congruence criterion?

**b.** State what you think the SAS criterion would say.

**c.** Use geometry software to construct the following triangles with the given measurements.

<table>
<thead>
<tr>
<th>( \triangle ABC )</th>
<th>( \triangle DEF )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( AB = 4 )</td>
<td>( DE = 2 )</td>
</tr>
<tr>
<td>( BC = 6 )</td>
<td>( EF = 3 )</td>
</tr>
<tr>
<td>( m\angle B = 47^\circ )</td>
<td>( m\angle E = 47^\circ )</td>
</tr>
</tbody>
</table>

**d.** Use the software to determine the remaining side and angle measures for each triangle.

**e.** Are the triangles similar? Explain why or why not.

**f.** Based on your answers above, state the SAS Similarity Theorem.
6. Now investigate whether there is an SSS similarity criterion.
   a. How do you think the SSS similarity criterion differs from the SSS congruence criterion?

   b. State what you think the SSS criterion would say.

   c. Use geometry software to construct the following triangles with the given measurements.

      \[ \triangle ABC \quad \triangle DEF \]
      \[
      \begin{array}{c|c}
        & \triangle ABC & \triangle DEF \\
        AB & 6 & DE \\
        BC & 8 & EF \\
        CA & 10 & FD \\
      \end{array}
      \]

   d. What are the angle measures of each triangle?

   e. Are the triangles similar? Explain why or why not.

   f. Based on your answers above, state the SSS Similarity Theorem.

7. For each pair of triangles, write the similarity criterion, if any, that can be used to show the triangles are similar.
   a. 
      \[
      \begin{array}{c}
        \triangle ABC \\
        10 \\
        38^\circ \\
        18 \\
      \end{array}
      \]
      \[
      \begin{array}{c}
        \triangle DEF \\
        9 \\
        38^\circ \\
        5 \\
      \end{array}
      \]
   
   b. 
      \[
      \begin{array}{c}
        \triangle ABC \\
        9 \\
        6 \\
        9 \\
      \end{array}
      \]
      \[
      \begin{array}{c}
        \triangle DEF \\
        6 \\
        6 \\
      \end{array}
      \]
   
   c. 
      \[
      \begin{array}{c}
        \triangle ABC \\
        21 \\
        85^\circ \\
      \end{array}
      \]
      \[
      \begin{array}{c}
        \triangle DEF \\
        14 \\
        85^\circ \\
      \end{array}
      \]
Lesson 18-1
Similarity Criteria

Check Your Understanding

8. Are all right isosceles triangles similar? Explain.

LESSON 18-1 PRACTICE

For each pair of triangles, write which similarity criterion, if any, can be used to show the triangles are similar. Remember to use complete sentences and words such as and, or, since, for example, therefore, because of, by the, to make connections between your thoughts.

10. 11. 12. 13.

14. Make use of structure. What additional information do you need, if any, in order to conclude that $\triangle ACD \sim \triangle BCD$? Is there more than one set of information that would work? Explain.
Clarissa is one of the surveyors hired to help determine land boundaries. She needs to find the distance across a ravine. She thinks she can use properties of similar triangles to find the distance. She locates points A, B, C, D, and E and takes the measurements shown.

1. Make sense of problems. Which similarity criterion, if any, can be used to show $\triangle ABC \sim \triangle DEC$? Explain.

2. The triangles are similar, so the corresponding sides are proportional. Complete and solve the proportion to find the distance $AB$.

$$\frac{65}{\phantom{65}} = \frac{\phantom{65}}{AB}$$

Clarissa measured the length of the ravine indirectly. **Indirect measurement** is a technique that does not use a measurement tool, such as a ruler, to measure an unknown length. Indirect measurement is used to measure distances that cannot be measured directly, such as the height of a tree or the distance across a lake or a ravine. Triangle similarity is often used to make indirect measurements.
Example A

Clarissa needs to find the width of a rock formation in the national park. She locates points $A$, $B$, $C$, $D$, and $E$ and takes the measurements shown. Find the width of the rock formation, $AB$.

Step 1: Show $\triangle DEC \sim \triangle BEA$.

Find $CE$ and $DE$.

$CE = 84 \text{ ft} - 36 \text{ ft} = 48 \text{ ft}$

$DE = 98 \text{ ft} - 42 \text{ ft} = 56 \text{ ft}$

Write a proportion to determine whether corresponding sides are proportional.

$$\frac{CE}{AE} = \frac{DE}{BE}$$

$$\frac{48}{84} = \frac{56}{98}$$

$48 \cdot 98 = 56 \cdot 84$

$4704 = 4704 \checkmark$

By the Reflexive Property, $\angle E \cong \angle E$. So by SAS similarity, $\triangle DEC \sim \triangle BEA$.

Step 2: Write and solve a proportion to find $AB$.

$$\frac{CE}{AE} = \frac{CD}{AB}$$

$$\frac{48}{84} = \frac{54}{AB}$$

$48 \cdot AB = 54 \cdot 84$

$AB = 94.5$

Solution: The width of the rock formation is 94.5 feet.
Lesson 18-2
Using Similarity Criteria

Try These A

a. Find $CB$.

b. Find $XY$.

Check Your Understanding

3. Jorge found $DA$ by writing and solving the proportion shown, but he is incorrect. What mistake did Jorge make?

$$\frac{10}{12} = \frac{DB}{DA}$$
$$10 \cdot DA = 12 \cdot 12$$
$$DA = 14.4$$

LESSON 18-2 PRACTICE

4. Determine whether each pair of triangles is similar. If so, write the similarity criterion that can be used to show they are similar and find the unknown measure.

a.

b.

5. Critique the reasoning of others. $\triangle ABC \sim \triangle DEC$. Clarissa wants to measure another ravine and wrote the following proportion to find $EC$. Is she correct? If not, explain why and correct her work. Then find $EC$.

$$\frac{9}{10} = \frac{8.75}{EC}$$
Learning Targets:
- Prove the Triangle Proportionality Theorem and its converse.
- Apply the Triangle Proportionality Theorem and its converse.

SUGGESTED LEARNING STRATEGIES: Close Reading, Marking the Text, Questioning the Text, Think-Pair-Share, Create Representations, Identify a Subtask, Work Backward

Clarissa needs to find the distance across a lake in the national park. She locates points \( A, B, C, D, \) and \( E \) and takes the measurements shown. She thinks she can use similar triangles to find the distance.

1. If \( MR \parallel ST \) in the figure below, explain why \( \triangle MAR \sim \triangle SAT \).

```
\[ \frac{AM}{AS} = \frac{AR}{?} \]
```

2. Knowing that corresponding sides of similar triangles are proportional, complete this proportion:

3. If \( AM = 12 \text{ cm}, MS = 9 \text{ cm}, \) and \( AR = 15 \text{ cm}, \) determine \( RT \). Show your work.

4. If \( MS = 8 \text{ cm}, AR = 25 \text{ cm}, \) and \( RT = 10 \text{ cm}, \) determine \( AM \). Show your work.
5. Complete the proof of the **Triangle Proportionality Theorem**.

Given: __________

Prove: \( \frac{b}{a} = \frac{c}{d} \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \triangle PIE ) and ( \triangle RIM ) with</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>2. If two parallel lines are cut by a transversal, the corresponding angles are congruent.</td>
</tr>
<tr>
<td>3.</td>
<td>3.</td>
</tr>
<tr>
<td>4. ( \triangle ) ____ ( \sim ) ( \triangle PIE )</td>
<td>4.</td>
</tr>
<tr>
<td>5. ( \frac{b}{b + a} = \frac{c}{c + d} )</td>
<td>5.</td>
</tr>
<tr>
<td>6. ( \frac{b(b + a)(c + d)}{(b + a)} = \frac{(b + a)(c + d)c}{(c + d)} )</td>
<td>6. Multiplication Property of Equality</td>
</tr>
<tr>
<td>7. ( b(c + d) = (b + a)c )</td>
<td>7. Property of the Multiplicative Identity</td>
</tr>
<tr>
<td>8.</td>
<td>8. Distributive Property</td>
</tr>
<tr>
<td>9. ( bd = ac )</td>
<td>9.</td>
</tr>
<tr>
<td>10. ( \frac{bd}{ad} = \frac{ac}{ad} )</td>
<td>10.</td>
</tr>
<tr>
<td>11.</td>
<td>11.</td>
</tr>
</tbody>
</table>

**MATH TERMS**

**Triangle Proportionality Theorem**
If a line parallel to a side of a triangle intersects the other two sides, then it divides them proportionally.
Lesson 18-3
Triangle Proportionality Theorem

Now you can use the theorem to solve the problem from the beginning of the lesson.

6. Clarissa needs to find the distance across the lake shown. The known measurements are shown. What is the distance $DA$?

7. State the converse of the Triangle Proportionality Theorem.

8. Construct viable arguments. Write a convincing argument about why the converse of the Triangle Proportionality Theorem is true.

Check Your Understanding

Consider $\triangle SAT$.

9. Determine if each statement is true or false.
   a. $\frac{AM}{MS} = \frac{AR}{RT}$
   b. $\frac{AM}{AS} = \frac{AR}{RT}$

10. If $AM = 8$ in., $AR = 12$ in., and $RT = 5$ in., what is $MS$?
11. Given: $RE \parallel AT \parallel IO \parallel NS$. Determine each length. Show your work.

### MATH TERMS

**Parallel Proportionality Theorem**
If two or more lines parallel to a side of a triangle intersect the other two sides of the triangle, then they divide them proportionally.

### MATH TIP

The Parallel Proportionality Theorem is a corollary of the Triangle Proportionality Theorem because it is proven directly from it.

#### ACTIVITY 18

**Lesson 18-3**

**Triangle Proportionality Theorem**

Given: $RE \parallel AT \parallel IO \parallel NS$. Determine each length. Show your work.

\[
\begin{align*}
C & \quad 36 \text{ cm} & R & \quad 24 \text{ cm} & A & \quad 27 \text{ cm} \\
E & \quad 48 \text{ cm} & T & \quad 40 \text{ cm} & N & \quad 12 \text{ cm} \\
S & & & & & \\
\end{align*}
\]

a. $ET$

b. $AI$

c. $AT$

d. $OS$

e. $IO$

f. $NS$
12. **Attend to precision.** A land developer is using a surveyor to measure distances to ensure that the streets in the new community are parallel.

![Diagram of streets with measurements](image)

a. If Grant Street and Newton Street are parallel, what is the value of x? Support your answer.

b. Are Smith Street and Grant Street parallel? Support your answer.

---

**Check Your Understanding**

13. Given the diagram with $LD || AE || NT$ and segment measures as shown, determine the following measures. Show your work.

![Diagram of segments](image)

- a. $SL$
- b. $LD$
- c. $ET$
- d. $NT$
LESSON 18-3 PRACTICE

14. State the Triangle Proportionality Theorem and its converse as a biconditional statement.

15. Describe the connection between the Triangle Midsegment Theorem and the Triangle Proportionality Theorem.

16. Reason abstractly. Given the diagram, determine whether the segments are parallel. Show your work.

- \( \overline{BJ} \) and \( \overline{CH} \)
- \( \overline{BJ} \) and \( \overline{EF} \)
- \( \overline{DG} \) and \( \overline{EF} \)
- \( \overline{BJ} \) and \( \overline{DG} \)
ACTIVITY 18 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 18-1

1. For each pair of triangles, write which similarity criterion, if any, can be used to show the triangles are similar.

   a.
   \[
   \begin{array}{c}
   24 \text{ cm} \\
   12 \text{ cm} \\
   36 \text{ cm}
   \end{array}
   \quad
   \begin{array}{c}
   12 \text{ cm} \\
   8 \text{ cm} \\
   4 \text{ cm}
   \end{array}
   \]

   b.
   \[
   \begin{array}{c}
   14 \text{ m} \\
   5 \text{ m}
   \end{array}
   \quad
   \begin{array}{c}
   14 \text{ m} \\
   5 \text{ m}
   \end{array}
   \]

   c.
   The arrows indicate parallel lines.

   d.
   \[
   \begin{array}{c}
   A \\
   6' \\
   B \\
   4' \\
   C \\
   9'
   \end{array}
   \quad
   \begin{array}{c}
   D \\
   6' \\
   E
   \end{array}
   \]

   e.
   \[
   \begin{array}{c}
   Y \\
   10 \text{ cm} \\
   X \\
   14 \text{ cm} \\
   Z
   \end{array}
   \quad
   \begin{array}{c}
   N \\
   7 \text{ cm} \\
   M \\
   5 \text{ cm} \\
   L
   \end{array}
   \]

Lesson 18-2

2. Solve for \(x\) in the following figure.

   \[
   \begin{array}{c}
   22 \text{ ft.} \\
   18 \text{ ft.}
   \end{array}
   \quad
   \begin{array}{c}
   4x + 1 \\
   7.5 \text{ ft.}
   \end{array}
   \]

3. The following triangles are similar. Determine the values of \(x\), \(y\), and \(z\).

   \[
   \begin{array}{c}
   15 + x \\
   20 \text{ m}
   \end{array}
   \quad
   \begin{array}{c}
   30 \text{ m}
   \end{array}
   \quad
   \begin{array}{c}
   z - 18^\circ \\
   24 \text{ m}
   \end{array}
   \]

4. \(\triangle CLU \sim \triangle ELU\). Given \(\triangle CEL\) with measures as shown, determine \(x\). Show your work.

   \[
   \begin{array}{c}
   C \\
   x
   \end{array}
   \quad
   \begin{array}{c}
   U \\
   36 \text{ cm}
   \end{array}
   \quad
   \begin{array}{c}
   E \\
   42 \text{ cm}
   \end{array}
   \]

5. Explain why the following triangles are similar to each other.

   \[
   \begin{array}{c}
   4n \\
   y \\
   3n
   \end{array}
   \quad
   \begin{array}{c}
   x \\
   7n \\
   z
   \end{array}
   \quad
   \begin{array}{c}
   4 \\
   y \\
   3
   \end{array}
   \quad
   \begin{array}{c}
   x \\
   7 \\
   z
   \end{array}
   \]
6. The lengths of the corresponding sides of two similar triangles are 10, 15, and 20 and 15, 22.5, and \( x \). What is the value of \( x \)?
   A. 20  
   B. 25  
   C. 30  
   D. 40

7. Standing 8 feet from a puddle of water on the ground, Gretchen, whose eye height is 5 feet 2 inches, can see the reflection of the top of a flagpole. The puddle is 20 feet from the flagpole. How tall is the flagpole?

8. Write a convincing argument to explain why \( \triangle TUV \sim \triangle RSV \).

**Lesson 18-3**

Given: \( \overline{AT} \parallel \overline{EY} \parallel \overline{SB} \). Complete each proportion with the appropriate measure.

9. \( \frac{EA}{BE} = \frac{TY}{?} \)
10. \( \frac{AT}{?} = \frac{UA}{UB} \)
11. \( \frac{AB}{UA} = \frac{?}{TU} \)

Determine the following measures. Show your work.

12. \( \underline{IK} \)
13. \( \underline{IN} \)
14. \( \underline{IT} \)
15. \( \underline{TH} \)

16. Given the diagram with \( \overline{TB} \parallel \overline{ST} \parallel \overline{EN} \), explain how to demonstrate that \( \frac{ES}{SB} = \frac{NT}{TI} \).

**MATHEMATICAL PRACTICES**

Model with Mathematics

17. If you sketched triangle \( MPR \) and drew a line parallel to side \( RP \) that intersected side \( MR \) at point \( X \) and side \( MP \) at point \( Z \), name a pair of similar triangles formed. Write two different proportions to show the relationships of the line segments in your figure.
Phaedra has finally saved up enough money from her after-school job to buy a new computer monitor. She wants to buy a monitor that is similar to her current monitor. She does an online search and decides to compare the three models shown to her current monitor.

Current monitor:

New models:
- Model 1: 20-inch monitor
- Model 2: 26-inch monitor
- Model 3: 21-inch monitor

1. **a.** Which of the new models, if any, are similar to Phaedra’s current monitor? Justify your response.
   **b.** For those that are similar to her current monitor, describe the scale factor that maps her current monitor to the new model.

2. Phaedra wants to mount her new monitor in a different position on her wall.

Describe the similarity transformations that map her current monitor to her new one.
3. Phaedra decides to buy the 20-inch model, which replaces her old 15-inch monitor. She knows that screen size is measured diagonally, so the measurements would be as follows:

Write a convincing argument using similarity criteria that explains why the triangles are similar to each other.

### Scoring Guide

<table>
<thead>
<tr>
<th>Scoring Guide</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics Knowledge and Thinking (Items 1, 2, 3)</td>
<td>• Clear and accurate understanding of similar figures, scale factor, and transformations</td>
<td>• Adequate understanding of similar figures, scale factor, and transformations</td>
<td>• Partial understanding of similar figures, scale factor, and transformations</td>
<td>• Little or no understanding of similar figures, scale factor, and transformations</td>
</tr>
<tr>
<td>Problem Solving (Items 1, 2)</td>
<td>• An appropriate and efficient strategy that results in a correct answer</td>
<td>• A strategy that may include unnecessary steps but results in a correct answer</td>
<td>• A strategy that results in some incorrect answers</td>
<td>• No clear strategy when solving problems</td>
</tr>
<tr>
<td>Mathematical Modeling / Representations (Item 2)</td>
<td>• Clear and accurate understanding of representations of similarity transformations</td>
<td>• A functional understanding of representations of similarity transformations</td>
<td>• Partial understanding of representations of similarity transformations</td>
<td>• Little or no understanding of representations of similarity transformations</td>
</tr>
<tr>
<td>Reasoning and Communication (Items 1a, 3)</td>
<td>• Precise use of appropriate math terms and language to justify whether or not figures are similar</td>
<td>• Mostly correct use of appropriate math terms and language to justify whether or not figures are similar</td>
<td>• Misleading or confusing use of appropriate math terms and language to justify whether or not figures are similar</td>
<td>• Incomplete or inaccurate use of appropriate math terms and language to justify whether or not figures are similar</td>
</tr>
</tbody>
</table>
Learning Targets:
- Identify the relationships that exist when an altitude is drawn to the hypotenuse of a right triangle.
- Prove the Right Triangle Altitude Theorem.

SUGGESTED LEARNING STRATEGIES: Close Reading, Marking the Text, Questioning the Text, Think-Pair-Share, Create Representations, Visualization

You have investigated properties and relationships of sides and angles in similar triangles. In this section, you examine a special characteristic of right triangles.

Given the figure with right triangle \( \triangle MAE \), \( \overline{AN} \perp \overline{ME} \), \( m \angle M = 70^\circ \).

1. Determine these angle measures.
   \[ m \angle MAN = \quad m \angle EAN = \quad m \angle E = \]

2. Justify that \( \triangle MAN \sim \triangle AEN \).

3. The large triangle is also similar to the two smaller triangles. Complete the similarity statement, naming the large triangle appropriately.
   \[ \triangle MAN \sim \]

4. Name the type of special segment \( AN \) is in relation to \( \triangle MAE \).

5. Given \( AN = 9 \) in, and \( NE = 12 \) in. Use the Pythagorean Theorem and the properties of similar triangles to determine these segment lengths. Show your work.
   \[ AE = \quad MA = \quad MN = \]
6. **Construct viable arguments.** Complete the proof of the *Right Triangle Altitude Theorem*.

Given: \( \triangle YEA \) with right angle \( EAY \) and altitude \( AS \)

Prove: \( \triangle YEA \sim \triangle YAS \sim \triangle AES \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( \angle ESA ) and ( \angle ASY ) are right angles.</td>
<td>2.</td>
</tr>
<tr>
<td>3. ( \angle ESA \cong \angle ASY \cong \angle EAY )</td>
<td>3.</td>
</tr>
<tr>
<td>4. ( \angle Y \cong \angle Y )</td>
<td>4.</td>
</tr>
<tr>
<td>5. ( \triangle \sim \triangle )</td>
<td>5.</td>
</tr>
<tr>
<td>6. ( \angle E \cong \angle E )</td>
<td>6.</td>
</tr>
<tr>
<td>7. ( \triangle \sim \triangle )</td>
<td>7.</td>
</tr>
<tr>
<td>8. ( \triangle YEA \sim \triangle YAS \sim \triangle AES )</td>
<td>8.</td>
</tr>
</tbody>
</table>
Lesson 19-1  
The Right Triangle Altitude Theorem

Check Your Understanding

7. There are three right triangles in the figure below. Draw each triangle with the right angle in the lower left position.

8. In the figure, \( \triangle XYZ \) is a right triangle and \( \overline{YW} \) is an altitude to the hypotenuse. Suppose you know that \( m \angle X = 62^\circ \). What other angle in the figure must also measure \( 62^\circ \)? Why?

9. Given the figure with right triangle \( \triangle DEF \), \( \overline{FG} \perp \overline{DE} \), \( m \angle E = 25^\circ \), \( DF = 8 \), \( DE = 17 \).
   Complete the similarity statement.
   \( \triangle DEF \sim \triangle \) _______ \( \sim \triangle \) _______

10. Use the figure in Item 9 to find each of the following measures.
   a. \( m \angle D = \) _______
   b. \( m \angle DFG = \) _______
   c. \( m \angle GFE = \) _______
   d. \( FE = \) _______
   e. \( FG = \) _______

11. Make use of structure. Triangle \( HIJ \) is an isosceles right triangle. Make a conjecture about the two triangles formed by drawing the altitude to the hypotenuse. Write a paragraph proof to prove your conjecture.
Learning Targets:
- Identify the relationships that exist when an altitude is drawn to the hypotenuse of a right triangle.
- Apply the relationships that exist when an altitude is drawn to the hypotenuse of a right triangle.

SUGGESTED LEARNING STRATEGIES: Close Reading, Think-Pair-Share, Create Representations, Self Revision/Peer Revision, Visualization

1. Which two similar triangles allow us to show \( \frac{x}{t} = \frac{t}{z} \)?

2. Which two similar triangles allow us to show \( \frac{x}{u} = \frac{u}{p} \)?

The answers to Items 1 and 2 indicate two corollaries of the Right Triangle Altitude Theorem.

**Corollary 1:** When the altitude is drawn to the hypotenuse of a right triangle, the length of the altitude is the geometric mean between the segments of the hypotenuse.

**Corollary 2:** When the altitude is drawn to the hypotenuse of a right triangle, each leg is the geometric mean between the hypotenuse and the part of the hypotenuse that is adjacent to the leg.

3. In Item 2 we used properties of similar triangles to justify that \( \frac{x}{u} = \frac{u}{p} \). Solve the proportion for \( u \). Show your work.

4. **Attend to precision.** Write and solve the proportion that can be written from corollary 2. Use \( w \).
Lesson 19-2
The Geometric Mean

Example A
Given \( \triangle DEF \) with altitude \( FG \). Determine \( x \), if \( y = 9 \) and \( z = 36 \).

\[
\begin{align*}
\text{Step 1.} & \quad \text{Use the corollary above to determine the relationships.} \\
& \quad \text{The altitude is drawn to the hypotenuse of the right triangle. So, the altitude, } x, \text{ is the geometric mean between the segments of the hypotenuse, } y \text{ and } z. \\
\text{Step 2.} & \quad \text{Write an equation and simplify.} \\
& \quad \frac{y}{x} = \frac{x}{z} \quad \text{Definition of geometric mean} \\
& \quad x^2 = yz \\
& \quad x^2 = 9 \cdot 36 \quad \text{Substitute 9 for } y \text{ and 36 for } z. \\
& \quad x^2 = 324 \quad \text{Multiply.} \\
& \quad \sqrt{x^2} = \sqrt{324} \\
& \quad x = 18 \quad \text{Take the square root of both sides.} \\
\end{align*}
\]

Try These A
Use the figure in Example A and the corollary above to answer Items a and b.

a. If \( y = 2 \) and \( x = 16 \), determine \( z \). Show your work.

b. If \( x = 18 \) and \( z = 45 \), determine \( y \). Show your work.

MATH TERMS
The geometric mean of two positive numbers \( a \) and \( b \) is the positive number \( x \) such that

\[
\frac{a}{x} = \frac{x}{b}. 
\]

When solving, \( x = \sqrt{ab} \).

Example:
6 is the geometric mean of 4 and 9 since \( \frac{4}{x} = \frac{x}{9} \), so that \( x^2 = 4(9) \) or \( x = \sqrt{4(9)} = 6 \).
Lesson 19-2
The Geometric Mean

Check Your Understanding

5. Explain how to use the corollary on the first page of this lesson to find JM. Then find JM.

6. Given \(\triangle AYE\) with altitude \(\overline{AS}\) (not to scale) as shown, solve the following. Show your work.
   a. \(x = 4\) cm, \(z = 9\) cm. Determine \(t\).
   b. \(x = 6\) cm, \(t = 12\) cm. Determine \(z\).
   c. \(z = 18\) cm, \(t = 32\) cm. Determine \(x\).

7. Explain how to use the two corollaries you learned in this lesson to find \(PJ\). Then find \(PJ\).

LESSON 19-2 PRACTICE

8. Find the geometric mean of each set of numbers. Write in radical form.
   a. 25 and 4
   b. 3 and 60
   c. 2.5 and 9.1

9. Given right triangle \(XYZ\), with altitude \(\overline{WZ}\), find each length.
   a. If \(XW = 3\) and \(WY = 12\), determine \(WZ\).
   b. If \(WZ = 4\) and \(XW = 2\), determine \(WY\).
   c. If \(XY = 22\) and \(XW = 9\), determine \(WZ\).
   d. If \(XZ = 10\) and \(XW = 4\), determine \(XY\).
   e. If \(WZ = 18\) and \(XW = 9\), determine \(XZ\).
   f. If \(WZ = 4\), \(XW = x\), and \(XY = 10\), determine \(ZY\).

10. Make sense of problems. Explain how the corollaries in this lesson can help you find the area of \(\triangle PQR\). What is the area of \(\triangle PQR\)?

\[\text{Area of } \triangle PQR = \frac{1}{2} \times \text{base} \times \text{height}\]
ACTIVITY 19 PRACTICE
Write your answers on notebook paper.
Show your work.

Lesson 19-1
1. Given right triangle $ABC$ with altitude $CE$. By the Right Triangle Altitude Theorem, which similarity statement is true?

   A. $\triangle ABC \sim \triangle EBC$
   B. $\triangle ABC \sim \triangle CBE$
   C. $\triangle ABC \sim \triangle AEC$
   D. $\triangle ABC \sim \triangle BCE$

2. Write a similarity statement comparing the three triangles in each figure.
   a. 
   
   b. 
   
   c. 

Lesson 19-2
3. Determine the geometric mean of 21 and 84.
4. Determine the geometric mean of 15 and 100.
5. The figure below shows a side view of a garage. The roof hangs over the base portion by 2 feet in the back. Determine each of the following.
   a. $x$
   b. $z$
   c. $y$
6. Given \( \triangle KID \) as shown.
   a. Determine \( ND \).
   b. Determine \( KD \).
   c. Determine \( KI \).

7. Given the kite with diagonal measures as shown.
   a. Determine the length of the short diagonal.
   b. Determine the side lengths.

8. Carly is 5 feet tall. She wants to know the height of a tree in her yard. She stands so that her lines of sight to the top of the tree and the bottom of the tree form a \( 90^\circ \) angle, as shown in the diagram below. How tall is the tree?

9. Juan built the birdhouse as shown. How tall is the roof of the birdhouse?

10. The altitude to the hypotenuse of a right triangle divides the hypotenuse into two segments. One segment is three times as long as the other. If the altitude is 12 mm long, what are the lengths of the two segments?

MATHEMATICAL PRACTICES
Reason Abstractly and Quantitatively

11. Given \( \triangle YEA \) with altitude \( AS \) as shown; if given any two of the variable measures, is it possible to determine all the other measures? If so, explain how. If it is not always possible, state when it is and when it is not.
The Pythagorean Theorem and Its Converse

Is That Right?
Lesson 20-1 Pythagorean Theorem

Learning Targets:
• Use similar triangles to prove the Pythagorean Theorem.
• Apply the Pythagorean Theorem to solve problems.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Summarizing, Paraphrasing, Think-Pair-Share, Create Representations

Sara owns an online company that manufactures custom kites. Her customers go to her website and design their own kites. Then Sara’s company builds them. Many of her customers don’t know the correct names for the parts of a kite. Sara is creating a Web page to educate them so they can communicate better with Sara and her staff.

Because many kites include right triangles, the Pythagorean Theorem is useful in analyzing the dimensions of a kite.

1. **Model with mathematics.** The simplest kite is the diamond kite shown. When the sides of this type of kite meet at a right angle, you can use the Pythagorean Theorem to find the length of the spine.
   a. Which side of triangle XYZ is the hypotenuse? Which sides are the legs?
   b. What is the length of the spine? Show your work.

One way to prove the Pythagorean Theorem is by using similar triangles. In right triangle ABC below, an altitude is drawn to hypotenuse AB, forming two right triangles that are similar to triangle ABC.

2. Write a similarity statement for the three similar triangles.

*Suggested Learning Strategies:*
Marking the Text, Summarizing, Paraphrasing, Think-Pair-Share, Create Representations
3. The corresponding sides of similar triangles are proportional, so you can write proportions involving sides of the triangles from Item 2. Complete each proportion. Then find the cross products.

\[
\frac{b}{x} = \frac{c}{b} \quad \frac{a}{a} = \frac{c}{c}
\]

4. Use the equations from Item 3 and algebra to prove \(a^2 + b^2 = c^2\).

**Check Your Understanding**

5. A **Pythagorean triple** is a set of three nonzero whole numbers that satisfy the Pythagorean Theorem. Explain why the numbers 3, 4, and 6 do not form a Pythagorean triple.

6. Explain why the Pythagorean Theorem relationship, \(a^2 + b^2 = c^2\), is only true if \(c\) is greater than both \(a\) and \(b\).

One of Sara’s customers designed the rhombus-shaped kite shown.

7. The length of the spine, \(AC\), is 28 inches, and the length of the spar, \(DB\), is 24 inches.

a. **Construct viable arguments.** Explain how to find the perimeter of a kite if the lengths of the spine and the spar are known. Include the properties of a rhombus in your explanation.

b. Use the Pythagorean Theorem to find \(AD\). Then find the perimeter of the kite.
Lesson 20-1
Pythagorean Theorem

Check Your Understanding

8. Use the Pythagorean Theorem to show the diagonals of a square with side length \( s \) are congruent.
9. How high up a vertical wall will a 24-foot ladder reach if the foot of the ladder is placed 10 feet from the wall? Show your work.

LESSON 20-1 PRACTICE

10. Find each unknown length. Simplify your answer in radical form.
   a. \[ b = \]
      \[
      \begin{array}{c}
      \text{27}
      \\
      \text{38}
      \\
      \hline
      b
      \end{array}
      \]
   b. \[ x = \]
      \[
      \begin{array}{c}
      \text{35}
      \\
      \text{14}
      \\
      \hline
      x
      \end{array}
      \]

11. Reason quantitatively. Find the area of a rectangular rug if the width of the rug is 13 feet and the diagonal measures 20 feet.

12. Which of the following is a Pythagorean triple?
   A. 3, 4, 6
   B. 7, 25, 26
   C. 15, 21, 25
   D. 9, 40, 41

13. One of the diagonals of a rectangle measures 15 cm. The width of the rectangle is 6 cm. Determine the perimeter of the rectangle.

14. The longer diagonal of a rhombus is 16 cm. Determine the length of the shorter diagonal, if the perimeter of the rhombus is 40 cm.

15. Construct viable arguments. To store his art supplies, Kyle buys a cube-shaped box with 8-inch sides. His longest paintbrush is 13.5 inches long. Explain how Kyle determined that the 13.5-inch paintbrush could fit in the box.
Learning Targets:
- Use the converse of the Pythagorean Theorem to solve problems.
- Develop and apply Pythagorean inequalities.

SUGGESTED LEARNING STRATEGIES: Use Manipulatives, Look for a Pattern, Create Representations, Activating Prior Knowledge, Think-Pair-Share

One of Sara’s customers designed the kite shown. The customer claims that \( \triangle QRS \) is a right triangle.

1. a. Write the Pythagorean Theorem in if-then form.

b. The converse of the Pythagorean Theorem is also true. Write the converse of the Pythagorean Theorem in if-then form.

c. Use the converse of the Pythagorean Theorem to determine whether \( \triangle QRS \) is a right triangle.

2. Construct viable arguments. Write an algebraic proof to show that if \( a, b, \) and \( c \) form a Pythagorean triple, then any positive whole-number multiple of the numbers is also a Pythagorean triple.

Given: Positive whole numbers \( a, b, \) and \( c \) with \( a^2 + b^2 = c^2 \)

Prove: For any positive whole number \( x, (xa)^2 + (xb)^2 = (xc)^2 \).
Lesson 20-2
Converse of the Pythagorean Theorem

3. Use each of the following sets of triangle side lengths to build triangles using the manipulatives (straws) provided by your teacher.

   **Step 1:** Cut straws into 5 cm, 6 cm, 12 cm, 13 cm, and 15 cm lengths.
   **Step 2:** Build each triangle on centimeter grid paper.
   **Step 3:** Identify each triangle as right, acute, or obtuse.
   **Step 4:** Complete the table.

<table>
<thead>
<tr>
<th>Triangle Side Lengths</th>
<th>Type of Triangle</th>
<th>Square of Longest Segment</th>
<th>Sum of the Squares of the Two Shorter Sides</th>
</tr>
</thead>
<tbody>
<tr>
<td>5, 12, 13</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6, 6, 12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5, 6, 12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5, 12, 15</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5, 12, 12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6, 12, 13</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6, 12, 15</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. **Express regularity in repeated reasoning.** If \(a\) and \(b\) represent the legs and \(c\) represents the hypotenuse, what do the results in Item 3 suggest about how \(a^2 + b^2\) compares to \(c^2\) for the different types of triangles?

5. Use the converse of the Pythagorean Theorem to determine whether each of the following sets of side lengths forms a right triangle. If a right triangle is not possible, tell whether an acute or obtuse triangle can be formed. Show the method you use to determine your answers.
   a. 12, 34, 37
   b. \(\frac{6}{7}, \frac{8}{7}, \frac{10}{7}\)
   c. 20, \(\sqrt{42}\), 21

6. Sara’s best customer wants to design a kite in the shape of a triangle with side lengths 27 inches, 34 inches, and 45 inches. Classify the triangle.
Lesson 20-2
Converse of the Pythagorean Theorem

Check Your Understanding

7. Determine whether the triangle below is right, acute, or obtuse. Justify your reasoning.

[Diagram of a triangle with sides 48, 73, and 54]

8. The numbers 11, 60, and 61 form a Pythagorean triple. Use this fact to write two additional Pythagorean triples.

LESSON 20-2 PRACTICE

9. Tell whether each triangle is a right triangle. Justify your reasoning.
   a. [Diagram of a triangle with sides 16, 63, and 65]
   b. [Diagram of a triangle with sides 24, 25, and 10]

10. Tell whether a triangle having the following side lengths can be formed. If a triangle can be formed, tell whether it is right, acute, or obtuse.
    a. 36, 77, 85
    b. 22, 18, 3
    c. 33, 56, 68
    d. 6, 11, 12

11. Use appropriate tools strategically. Shauntay is making a picture frame. Explain how Shauntay can use a ruler and the converse of the Pythagorean Theorem to determine whether the sides meet at right angles.

[Diagram of a rectangle with corners labeled A, B, C, and D]
**ACTIVITY 20 PRACTICE**

Write your answers on notebook paper or on grid paper. Show your work.

**Lesson 20-1**

1. Find the length of the hypotenuse of each right triangle with the given leg lengths. Express the answer as a simplified radical.
   a. legs: 11 ft and 60 ft
   b. legs: 7 mm and 8 mm
   c. legs: 40 cm and 41 cm
   d. legs: 20 in. and 99 in.
   e. legs: 16 ft and 23 ft

2. Find each unknown length. Express the length in radical form.
   a. \[ x = \sqrt{3^2 - 4^2} \]
   b. \[ y = \sqrt{11^2 - 18^2} \]
   c. \[ x = \sqrt{33^2 - 52^2} \]

3. Find the area of parallelogram \(ABCD\).

4. If a flat-screen television is a rectangle with a 53-inch diagonal and a width of 45 inches, what is the height of the screen?

5. A standard baseball diamond is a square 90 feet on each side. Find the distance of a throw made from the catcher 3 feet behind home plate in an attempt to throw out a runner trying to steal second base. Round to the nearest whole number.
   A. 93 ft  
   B. 124 ft  
   C. 130 ft  
   D. 183 ft

6. Which best approximates the lengths of the legs of a right triangle if the hypotenuse is 125 mm and the shorter leg is one-half the length of the longer leg?
   A. 25 mm and 55 mm  
   B. 56 mm and 112 mm  
   C. 5 mm and 10 mm  
   D. 63 mm and 63 mm

7. A kite is shaped like an isosceles triangle.

   To the nearest tenth, what is the length of the spine?

8. Find the length of the hypotenuse of an isosceles right triangle with leg length 5 centimeters. Give the exact answer.

9. Find the length of the altitude drawn from the vertex of an isosceles triangle with side lengths 13 in., 13 in., and 24 in.

10. An isosceles trapezoid has bases that are 7 inches and 13 inches long. The height of the trapezoid is 4 inches. Find the perimeter of the trapezoid.
11. Tell whether each triangle is a right triangle.
   a. \[65 \quad 97 \quad 72\]
   b. \[9 \quad 40 \quad 42\]
   c. \[44 \quad 117 \quad 125\]
   d. \[58 \quad 45 \quad 40\]

12. Tell whether a triangle having the following side lengths can be formed. If a triangle can be formed, tell whether it is right, acute, or obtuse.
   a. 9, 40, 41
   b. 4, 5, 6
   c. 5, 12, 18
   d. 9, 9, 13
   e. 27, 36, 45
   f. \[\sqrt{8}, \sqrt{8}, \sqrt{16}\]

13. Use the given vertices to determine whether \(\triangle ABC\) is a right triangle. Explain your reasoning and show the calculations that led to your answer. \(A(2, 7), B(3, 6), C(-4, -1)\).

14. A triangle has side lengths of \(x, 2x,\) and 45. If the length of the longest side is 45, what values of \(x\) make the triangle acute? right? obtuse?

15. Which of the following cannot be the side lengths of a triangle?
   A. 10, 12, 18
   B. 9, 9, 10
   C. 3, 11, 15
   D. 10, 15, 20

16. Complete the following proof.
   **Given:** \(\triangle ABC, c^2 > a^2 + b^2\), where \(c\) is the length of the longest side of the triangle, and right \(\triangle ABX\) with side lengths \(a, b,\) and \(x\), where \(x\) is the length of the hypotenuse.
   **Prove:** \(\triangle ABC\) is obtuse.

   \[
   \begin{align*}
   1. \quad \triangle ABC, c^2 > a^2 + b^2, \text{ where } c \text{ is the length of the longest side of the triangle, and right } \triangle ABX \text{ with side lengths } a, b, \text{ and } x, \text{ where } x \text{ is the length of the hypotenuse.} \\
   2. \quad x^2 = a^2 + b^2 \\
   3. \quad c^2 > _____ \\
   4. \quad c > _____ \\
   5. \quad m\angle X = 90^\circ \\
   6. \quad m\angle C > m_____ \\
   7. \\
   8. \quad C \text{ is } _____ \\
   9. \quad \triangle ABC \text{ is } _____
   \end{align*}
   \]

**MATHEMATICAL PRACTICES**

**Look For and Make Use of Structure**

17. The Pythagorean Theorem was thought of by the early Greeks as the following: The area of the square on the hypotenuse of a right triangle is equal to the sum of the areas of the squares on the legs.

   Draw a diagram to illustrate this statement. Explain how your diagram illustrates the Pythagorean Theorem.
Learning Targets:

- Describe the relationships among the side lengths of 45°-45°-90° triangles.
- Apply relationships in special right triangles to solve problems.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Summarizing, Paraphrasing, Create Representations, Use Manipulatives, Predict and Confirm, Look for a Pattern, Quickwrite

The Community Hospital wants to make its rooms more cheerful. The hospital asked volunteers to sew quilts for patient rooms and to decorate the common areas. The Hoover High Student Council wants to participate in the project. Ms. Jones, a geometry teacher, decides to have her classes investigate the mathematical patterns found in quilts.

Quilt blocks are often squares made up of smaller fabric pieces sewn together to create a pattern. There are many different quilt block designs. Often these designs are named. A quilt block design made up of nine small squares, called the “Friendship Star,” is shown.

1. The Friendship Star quilt block contains five small squares and eight triangles.
   a. Identify congruent figures in the quilt block and explain why they are congruent.

b. Classify the triangles in the quilt block by their angle measures.

c. Classify the triangles in the quilt block by their side lengths.
d. What are the measures of the acute angles in each of the triangles? Explain your reasoning.

The Hoover High Student Council decided to make Friendship Star quilts of various sizes with different-sized quilt blocks.

2. Work with your group and use the Pythagorean Theorem to find the exact values of the missing dimensions and ratios in the table below.

<table>
<thead>
<tr>
<th>Dimensions of Quilt Block</th>
<th>Length of Triangle Leg (in inches)</th>
<th>Length of Hypotenuse (in inches)</th>
<th>Ratio of Hypotenuse to Leg</th>
</tr>
</thead>
<tbody>
<tr>
<td>9 in. × 9 in.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15 in. × 15 in.</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.5</td>
<td>3/2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6\sqrt{2}</td>
<td>4</td>
</tr>
</tbody>
</table>

3. What patterns do you notice in the table in Item 2?

4. Suppose that you are only given the length of one leg of an isosceles right triangle (45°-45°-90°).
   a. Write a verbal rule for finding the lengths of the other two sides.

   b. Let $l$ be the length of the leg of any isosceles right triangle (45°-45°-90°). Use the Pythagorean Theorem to derive an algebraic rule for finding the length of the hypotenuse, $h$, in terms of $l$. 
Lesson 21-1
45°-45°-90° Triangles

Check Your Understanding

For each 45°-45°-90° triangle, find the unknown length, $x$.

5. \[ \triangle \]

6. \[ \triangle \]

7. \[ \triangle \]

8. Ms. Jones designed a variation on the Friendship Star quilt block, called the “Twisted Star.” Using the rules you derived in Item 4, what are the dimensions of the smallest triangle on the 12 in. $\times$ 12 in. quilt block shown below? Explain how you found your answer.

9. The perimeter of a square picture frame is 40 cm. Find the length of a diagonal of the frame.
LESSON 21-1 PRACTICE

Unless otherwise indicated, write all answers in simplest radical form.

10. For each 45°-45°-90° triangle, find a and b.
   a. [Diagram of a 45°-45°-90° triangle with sides labeled a and b, and hypotenuse labeled 3.]
   b. [Diagram of a 45°-45°-90° triangle with sides labeled a and b, and hypotenuse labeled $8\sqrt{2}$.]
   c. [Diagram of a 45°-45°-90° triangle with sides labeled a and b, and hypotenuse labeled $9\sqrt{15}$.

11. The length of each leg of an isosceles right triangle is 5.
   a. Find the perimeter of the triangle.
   b. Find the area of the triangle.

12. **Attend to precision.** Find the perimeter of a square, as a simplified radical, if the length of its diagonal is 14 inches.
Learning Targets:
- Describe the relationships among the side lengths of $30^\circ$-$60^\circ$-$90^\circ$ triangles.
- Apply relationships in special right triangles to solve problems.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Create Representations, Quickwrite, Think-Pair-Share, Look for a Pattern

Ms. Jones introduces her class to a quilt block in the shape of a hexagon. This block is formed from six equilateral triangles, divided in half along their altitudes. Ms. Jones’s students know that if an altitude is drawn in an equilateral triangle, two congruent triangles are formed. The resulting hexagonal quilt block is shown below.

1. What is the measure of each of the angles in any equilateral triangle?
2. What are the measures of each of the angles in the smallest triangles in the hexagonal quilt block shown above? Explain your answer.

3. The smallest triangles are special scalene right triangles. They are often called $30^\circ$-$60^\circ$-$90^\circ$ right triangles.
   a. How can you determine which leg is shorter and which leg is longer using the angles of the triangle?

   b. If each of the sides of the hexagonal quilt block is 4 inches, how long is the shorter leg in the $30^\circ$-$60^\circ$-$90^\circ$ right triangle? Explain your answer.

   c. What is the relationship between the length of the hypotenuse and the length of the shorter leg in a $30^\circ$-$60^\circ$-$90^\circ$ triangle? Explain your answer.
4. Look for patterns between the longer leg of the $30^\circ$-$60^\circ$-$90^\circ$ right triangle and the other sides, by completing the table below using the Pythagorean Theorem. Write each ratio in simplest radical form.

<table>
<thead>
<tr>
<th>Length of Hypotenuse (in inches)</th>
<th>Length of Shorter Leg (in inches)</th>
<th>Length of Longer Leg (in inches)</th>
<th>Ratio of Length of Longer Leg to Length of Shorter Leg</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. **Express regularity in repeated reasoning.** What patterns do you notice in the table in Item 4?

6. Write a verbal rule for finding the length of the longer leg of a $30^\circ$-$60^\circ$-$90^\circ$ right triangle if you are given the length of the shorter leg.

7. Write a verbal rule for finding the length of the hypotenuse of a $30^\circ$-$60^\circ$-$90^\circ$ right triangle if you are given the length of the shorter leg.

8. Let $s$ be the length of the shorter leg of any $30^\circ$-$60^\circ$-$90^\circ$ right triangle. Use the Pythagorean Theorem to derive an algebraic rule for finding the length of the longer leg, $l$, in terms of $s$.

9. One of the students in Ms. Jones’s class wrote the proportion shown to find the unknown leg length $x$ in the quilt patch. Did the student correctly find the length? If not, explain why and correct the error.

\[
\frac{x}{5} = \frac{\sqrt{3}}{1}
\]

\[
x = 5\sqrt{3}
\]
Lesson 21-2
30°-60°-90° Triangles

Check Your Understanding

10. In a 30°-60°-90° triangle, the length of the shorter leg is 6 in.
   a. What is the length of the longer leg?
   b. What is the length of the hypotenuse?

11. In a 30°-60°-90° triangle, the length of the hypotenuse is 10 cm.
   a. What is the length of the shorter leg?
   b. What is the length of the longer leg?

LESSON 21-2 PRACTICE

Unless otherwise indicated, write all answers in simplest radical form.

12. **Model with mathematics**. Use your work from Items 4–7 to determine the height of the hexagon block shown below, whose sides are 4 inches, if the height is measured from the midpoint of one side to the midpoint of the opposite side. Explain how you found your answer.

![Hexagon block diagram]

13. Find \(d\) and \(e\).

![Diagram with 60° angle and sides 8 and \(e\)]
14. The longer leg of a $30^\circ$-$60^\circ$-$90^\circ$ triangle is 6 inches. What is the length of the hypotenuse?
15. The length of an altitude of an equilateral triangle is $2\sqrt{3}$ inches. Find the length of a side of the triangle.
16. One side of an equilateral triangle is 8 cm. Find the length of the altitude.
17. The perimeter of an equilateral triangle is 36 inches. Find the length of an altitude.
18. Find the perimeter of the trapezoid.

![Trapezoid Diagram]

19. Find the perimeter of $\triangle PQR$.
ACTIVITY 21 PRACTICE

Write your answers on notebook paper. Show your work.

Unless otherwise indicated, write all answers in simplest radical form.

Lesson 21-1

1. In a 45°-45°-90° triangle, if the length of a leg is 6 cm, the length of the hypotenuse is:
   A. 12 cm
   B. $6\sqrt{3}$ cm
   C. $6\sqrt{5}$ cm
   D. $6\sqrt{2}$ cm

2. For each 45°-45°-90° triangle, find $a$ and $b$.
   a. 
   
   ![Diagram of a 45°-45°-90° triangle with side lengths](image)
   b. 
   
   ![Diagram of a 45°-45°-90° triangle with side lengths](image)
   c. 
   
   ![Diagram of a 45°-45°-90° triangle with side lengths](image)

3. Square $MNOP$ has a diagonal of 12 inches. Find the length of each side of the square.

4. The length of the hypotenuse of an isosceles right triangle is 8.
   a. Find the perimeter of the triangle.
   b. Find the area of the triangle.

5. Find the perimeter of a square, as a simplified radical, if the length of its diagonal is $4\sqrt{10}$ inches.

6. Which of the following statements is true?
   A. $BC = \sqrt{2}BA$
   B. $BA = BC$
   C. $BA = \sqrt{2}BC$
   D. $BC = 2BA$
Lesson 21-2

7. Find $a$ and $b$.

![Diagram of a 30°-60° right triangle with sides labeled $a$, $b$, and hypotenuse of 42 units.]

8. Find $a$ and $c$.

![Diagram of a 10°-60° right triangle with sides labeled $a$, $c$, and hypotenuse of 10 units.]

9. Find $m$.

![Diagram of an equilateral triangle with a side length of 8 units and an angle of 60°.]

11. An equilateral triangle has a side length of 4 ft. What is the area of the triangle?
   - A. $12\text{ ft}^2$
   - B. $4\sqrt{3}\text{ ft}^2$
   - C. $6\text{ ft}^2$
   - D. $2\sqrt{3}\text{ ft}^2$

12. What is the area of the quilt patch, as a simplified radical?

![Diagram of a quilt patch with sides labeled 8 inches and 6 inches.]

13. Find $a$ and $b$.

![Diagram of a right triangle with angles labeled 45° and 30°.]

14. Brayden's teacher asked him to draw a 30°-60°-90° right triangle. He drew the figure shown. Tell why it is not possible for Brayden's triangle to exist.

![Diagram of a hypothetical 30°-60°-90° right triangle with sides labeled 5, 12, and 13 units.]

MATHEMATICAL PRACTICES
Construct Viable Arguments and Critique the Reasoning of Others

14. Brayden's teacher asked him to draw a 30°-60°-90° right triangle. He drew the figure shown. Tell why it is not possible for Brayden's triangle to exist.
Yoshio owns a farm that uses wind turbines to produce wind power. Wind power, also known as wind energy, uses the power of the wind to generate electrical or mechanical power.

1. One of the wind turbines that Yoshio owns is shown in the figure below. He needs to know the height of the turbine. Yoshio is 5 feet 6 inches tall. If he stands 12 feet from the turbine, his line of sight forms a 90° angle with the top and bottom of the turbine.

   ![Diagram of a wind turbine with a person standing near it]

   a. What special segment is $ZW$? Explain.
   b. Write a similarity statement to show the relationship among triangle $XYZ$ and the two triangles formed by drawing $ZW$.
   c. What is the height of the turbine?

2. Another wind turbine that Yoshio owns uses guy wires for support. In the figure, $AC = 15$ ft, $\angle ABC = 60°$, $\angle ADC = 45°$, and $\angle AEC = 30°$.

   Find each of the following measures.
   a. $AB = \underline{\hspace{2cm}}$
   b. $AD = \underline{\hspace{2cm}}$
   c. $AE = \underline{\hspace{2cm}}$
   d. What is the height of the turbine, $EC$?

3. Use the converse of the Pythagorean Theorem to prove that triangle $AEC$ is a right triangle.
### Scoring Guide

<table>
<thead>
<tr>
<th>Mathematics Knowledge and Thinking (Items 1, 2, 3)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Clear and accurate understanding of similar figures created by an altitude drawn to the hypotenuse of a right triangle, special right triangles, and the Converse of the Pythagorean Theorem</td>
<td>• Adequate understanding of similar figures created by an altitude drawn to the hypotenuse of a right triangle, special right triangles, and the Converse of the Pythagorean Theorem</td>
<td>• Partial understanding of similar figures created by an altitude drawn to the hypotenuse of a right triangle, special right triangles, and the Converse of the Pythagorean Theorem</td>
<td>• Little or no understanding of similar figures created by an altitude drawn to the hypotenuse of a right triangle, special right triangles, and the Converse of the Pythagorean Theorem</td>
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</tr>
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</table>

<table>
<thead>
<tr>
<th>Problem Solving (Items 1, 2)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• An appropriate and efficient strategy that results in correct answers</td>
<td>• A strategy that may include unnecessary steps but results in correct answers</td>
<td>• A strategy that results in some incorrect answers</td>
<td>• No clear strategy when solving problems</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Mathematical Modeling / Representations (Items 1b, 1c)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Clear and accurate understanding of writing a similarity statement to show the relationship among similar triangles created by drawing an altitude to the hypotenuse of a right triangle</td>
<td>• Mostly accurate understanding of writing a similarity statement to show the relationship among similar triangles created by drawing an altitude to the hypotenuse of a right triangle</td>
<td>• Partial understanding of writing a similarity statement to show the relationship among similar triangles created by drawing an altitude to the hypotenuse of a right triangle</td>
<td>• Little or no understanding of writing a similarity statement to show the relationship among similar triangles created by drawing an altitude to the hypotenuse of a right triangle</td>
<td></td>
</tr>
<tr>
<td>• Clear and accurate understanding of creating a proportion to determine missing lengths in similar triangles</td>
<td>• A functional understanding of creating a proportion to determine missing lengths in similar triangles</td>
<td>• Partial understanding of creating a proportion to determine missing lengths in similar triangles</td>
<td>• Little or no understanding of creating a proportion to determine missing lengths in similar triangles</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reasoning and Communication (Item 3)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Precise use of appropriate mathematics and language to justify whether or not triangle AEC is a right triangle</td>
<td>• Mostly correct use of appropriate mathematics and language to justify whether or not triangle AEC is a right triangle</td>
<td>• Misleading or confusing use of appropriate mathematics and language to justify whether or not triangle AEC is a right triangle</td>
<td>• Incomplete or inaccurate use of appropriate mathematics and language to justify whether or not triangle AEC is a right triangle</td>
<td></td>
</tr>
</tbody>
</table>
Tricia is a commercial artist working for The Right Angle Company. The company specializes in small business public relations. Tricia creates appealing logos for client companies. In fact, she helped create the logo for her company. The Right Angle Company will use its logo in different sizes for stationery letterhead, business cards, and magazine advertisements. The advertisement and stationery letterhead-size logos are shown below.

1. Use appropriate tools strategically. Measure the two acute angles and the lengths of the sides of each logo above. Measure the angles to the nearest degree and the sides to the nearest tenth of a centimeter. Be as accurate as possible. Record the results in the table below.

<table>
<thead>
<tr>
<th>Logo Size</th>
<th>Hypotenuse</th>
<th>Longer Leg</th>
<th>Shorter Leg</th>
<th>Larger Acute Angle</th>
<th>Smaller Acute Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Advertisement</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Letterhead</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. Although the measurements can never be exact, the lengths of the sides of any right triangle satisfy the Pythagorean Theorem. Confirm that the Pythagorean Theorem is satisfied by the measurements of the two right triangular logos on the preceding page. Show your work and results but allow for some error due to measurement limitations.

3. The logos are similar triangles. Justify this statement. Then give the scale factor of advertising logo lengths to corresponding letterhead logo lengths.

4. The triangular logo used on The Right Angle Company business cards is also similar to the logos used for advertisements and letterheads. The scale factor of letterhead logo lengths to corresponding business card logo lengths is 2.3:1.
   a. Determine the length of each side of the business card logo.
   b. Use a ruler to draw the business card logo to scale in the space below.
Lesson 22-1
Similar Right Triangles

Tricia tries to incorporate a right triangle into many of the logos she designs for her clients. As she does, Tricia becomes aware of a relationship that exists between the measures of the acute angles and the ratios of the lengths of the sides of the right triangles.

5. Use each grid below to draw a right triangle that has a longer vertical leg of \( L \) units and a shorter horizontal leg of \( S \) units. The first triangle is drawn for you. Use the Pythagorean Theorem to find the length \( H \) of the resulting hypotenuse to the nearest tenth. Record its length in the appropriate place at the bottom of each grid.

<table>
<thead>
<tr>
<th>Grid A</th>
<th>Grid B</th>
<th>Grid C</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L = 8, S = 6 )</td>
<td>( L = 5, S = 4 )</td>
<td>( L = 4, S = 3 )</td>
</tr>
</tbody>
</table>

\[ H = \]

<table>
<thead>
<tr>
<th>Grid D</th>
<th>Grid E</th>
<th>Grid F</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L = 10, S = 8 )</td>
<td>( L = 12, S = 9 )</td>
<td>( L = 6, S = 3 )</td>
</tr>
</tbody>
</table>

\[ H = \]
6. Some of the triangles in Grids A–F are similar to each other. Identify the groups of similar triangles using the grid letter and explain below how you know they are similar. You should find a group of three similar triangles and a group of two similar triangles.

In any right triangle, the hypotenuse is opposite the right angle. For each acute angle, one of the right triangle’s legs is known as that angle’s **opposite leg** and the remaining leg is known as that angle’s **adjacent leg**. In \( \triangle CAR \) below, the hypotenuse is \( AC \). For acute \( \angle C \), side \( AR \) is its opposite leg and side \( RC \) is its adjacent leg. For acute \( \angle A \), side \( RC \) is its opposite leg and side \( AR \) is its adjacent leg.

7. In right \( \triangle BUS \), identify the opposite leg and the adjacent leg for \( \angle U \).
Lesson 22-1
Similar Right Triangles

Check Your Understanding

8. Are all isosceles right triangles similar? Explain.
9. Two right triangles are similar with a scale factor of 3:4.5. The triangle with the shorter hypotenuse has leg lengths of 6 and 8. What is the length of the longer hypotenuse?

LESSON 22-1 PRACTICE

10. Use \( \triangle QRS \) to find the following.
   a. the leg opposite \( \angle Q \)
   b. the leg adjacent to \( \angle Q \)
   c. the leg opposite \( \angle R \)
   d. the leg adjacent to \( \angle R \)
   e. the hypotenuse

11. Make sense of problems. Find the scale factor and the unknown side lengths for each pair of similar triangles.
   a. \( \triangle ABC \sim \triangle DEF \)
      Scale factor __________
      \( AC = \) __________
      \( AB = \) __________
      \( DE = \) __________

   b. \( \triangle TUV \sim \triangle XYZ \)
      Scale factor __________
      \( UT = \) __________
      \( YZ = \) __________
      \( XY = \) __________

   c. \( \triangle LMN \sim \triangle GHI \)
      Scale factor __________
      \( NL = \) __________
      \( HI = \) __________
      \( GI = \) __________
Learning Targets:

- Understand the definitions of sine, cosine, and tangent ratios.
- Calculate the trigonometric ratios in a right triangle.
- Describe the relationship between the sine and cosine of complementary angles.

SUGGESTED LEARNING STRATEGIES: Create Representations, Graphic Organizer, Look for a Pattern, Quickwrite, Interactive Word Wall

1. One group of similar triangles, identified in Item 5 of the previous lesson, is shown on the grids below. For each right triangle, the vertex opposite the longer leg has been named with the same letter as the grid. Determine the ratios in the table and write the ratios in lowest terms.

<table>
<thead>
<tr>
<th>Grid A</th>
<th>Grid C</th>
<th>Grid E</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L = 8, S = 6$</td>
<td>$L = 4, S = 3$</td>
<td>$L = 12, S = 9$</td>
</tr>
<tr>
<td>$H = 10$</td>
<td>$H = 5$</td>
<td>$H = 15$</td>
</tr>
</tbody>
</table>

**Table:**

<table>
<thead>
<tr>
<th>Triangle</th>
<th>Length of Opposite Leg</th>
<th>Length of Hypotenuse</th>
<th>Length of Adjacent Leg</th>
<th>Length of Hypotenuse</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\angle A$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\angle C$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\angle E$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Lesson 22-2
Trigonometric Ratios

2. For each of the triangles in Item 1, use your protractor to find the measure of the larger acute angle.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Measure of Larger Acute Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>∠A</td>
<td></td>
</tr>
<tr>
<td>∠C</td>
<td></td>
</tr>
<tr>
<td>∠E</td>
<td></td>
</tr>
</tbody>
</table>

3. In Item 2, you found that the measures of each of the three angles are the same. If, in another right triangle, the measure of the larger acute angle was the same as the measures of ∠A, ∠C, and ∠E, what would you expect the following ratios to be?
   a. length of opposite leg = length of hypotenuse
   b. length of adjacent leg = length of hypotenuse
   c. length of opposite leg = length of adjacent leg

4. Explain how you reached your conclusions in Item 3.

The ratio of the lengths of two sides of a right triangle is a trigonometric ratio. The three basic trigonometric ratios are sine, cosine, and tangent, which are abbreviated sin, cos, and tan.

5. Use appropriate tools strategically. Use a scientific or graphing calculator to evaluate each of the following to the nearest tenth. Make sure your calculator is in DEGREE mode.
   a. sin 53° =
   b. cos 53° =
   c. tan 53° =

6. In each column of the table in Item 1, the ratios that you wrote are equal. Express the ratios from the three columns as decimal numbers rounded to the nearest tenth.
   a. length of opposite leg = length of hypotenuse
   b. length of adjacent leg = length of hypotenuse
   c. length of opposite leg = length of adjacent leg
7. Compare your answers to Items 5 and 6. Then describe each of the
ratios below in terms of sin, cos, and tan. Assume that the ratios
represent sides of a right triangle in relation to acute \( \angle X \).

a. \[
\frac{\text{length of opposite leg}}{\text{length of hypotenuse}}
\]

b. \[
\frac{\text{length of adjacent leg}}{\text{length of hypotenuse}}
\]

c. \[
\frac{\text{length of opposite leg}}{\text{length of adjacent leg}}
\]

8. a. For \( \triangle ABC \), write the ratios in simplest form.

\[
\sin A = \quad \sin C =
\]

\[
\cos A = \quad \cos C =
\]

\[
\tan A = \quad \tan C =
\]

b. How are the two acute angles of \( \triangle ABC \) related?

c. What is the relationship between the sine and cosine of
complementary angles?

d. If you know \( \sin 68^\circ \approx 0.93 \), what other trigonometric ratio do you
know?
Lesson 22-2
Trigonometric Ratios

Check Your Understanding

9. Triangle \(ABC\) is a 30°-60°-90° triangle. Explain how to write \(\sin 30^\circ\) as a ratio in simplest form without knowing the length of any side of the triangle.

10. Suppose you know that \(\triangle RST\) is a right triangle with a right angle at \(\angle R\). If \(\cos S = 0.67\), what other trigonometric ratio can you write?

11. Given \(\sin B = \frac{5}{13}\), draw a right triangle \(ABC\) with right angle \(C\) and label the side lengths.
   a. Determine the length of the missing side.
   b. Determine \(\cos B\).
   c. What is \(\tan B\)?

LESSON 22-2 PRACTICE

12. Use a calculator to find each of the following. Round each value to the nearest hundredth.
    \[
    \begin{align*}
    \sin 48^\circ &= \underline{\text{______}} \\
    \tan 65^\circ &= \underline{\text{______}} \\
    \cos 12^\circ &= \underline{\text{______}} \\
    \sin 90^\circ &= \underline{\text{______}} \\
    \end{align*}
    \]

13. Write each ratio in simplest form.
    \[
    \begin{align*}
    \sin X &= \underline{\text{______}} \\
    \cos X &= \underline{\text{______}} \\
    \sin Y &= \underline{\text{______}} \\
    \cos Y &= \underline{\text{______}} \\
    \tan X &= \underline{\text{______}} \\
    \tan Y &= \underline{\text{______}} \\
    \end{align*}
    \]

14. Attend to precision. Elena was asked to write an explanation of how to find \(\tan P\). She wrote, “To find the tangent of \(\angle P\), I found the ratio of the length of the side opposite \(\angle P\) to the length of the side adjacent to \(\angle P\). Since these sides have the same length, \(\tan P = 1\) ft.” Critique Elena’s statement. Is there anything she should have written differently? If so, what?
Learning Targets:

- Use trigonometric ratios to find unknown side lengths in right triangles.
- Solve real-world problems using trigonometric ratios.

SUGGESTED LEARNING STRATEGIES: Quickwrite, Create Representations, Identify a Subtask

1. For each of the following triangles, determine the ratios requested. Then use a scientific or graphing calculator to evaluate each trigonometric function to the nearest thousandth and solve each equation for $y$. Round final answers to the nearest tenth.

   a. $P_1R_1y_1Q_1$
      
      \[
      \sin 41^\circ = \quad \cos 28^\circ =
      \]

   b. $M_1N_1O_1y_2$
      
      \[
      \sin 62^\circ = \quad \cos 28^\circ =
      \]

2. Use your knowledge of trigonometric functions to find the value of $x$ in $\triangle ABC$.
   
   a. Choose an acute angle in $\triangle ABC$.

   b. Identify sides as opposite, adjacent, or hypotenuse with respect to the acute angle chosen.

   c. Use the sides to choose an appropriate trigonometric function.

   d. Write an equation using the identified sides, acute angle, and trigonometric function chosen.

   e. Solve for $x$. 
Lesson 22-3
Using Trigonometric Ratios

3. Use your knowledge of trigonometric functions to find the value of \( y \) in the triangle below.

\[ \begin{array}{c}
D \\
E \\
\end{array} \begin{array}{c}
12 \\
10^\circ \\
\end{array} \begin{array}{c}
F \\
\end{array} \]

\( y \)

a. Choose an acute angle in \( \triangle DEF \) and identify sides as adjacent, opposite, or hypotenuse with respect to the angle you chose.

b. Use the sides to choose an appropriate trigonometric ratio and write an equation using the identified sides, acute angle, and trigonometric function. Then solve for \( y \).

4. **Make sense of problems.** Tricia did such an exceptional job creating logos that she was given the task of making a banner and representing her company at a job fair. When Tricia got to the job fair, she was relieved to see there was a ladder she could use to hang the banner. While Tricia waited for someone to help her, she leaned the 12-foot ladder against the wall behind the booth. The ladder made an angle of 75° with the floor.

a. Use the information above to draw and label a right triangle to illustrate the relationship between the ladder and the wall.

b. Set up and solve an equation to find how far up the wall the top of the ladder reaches.

c. Find the distance from the base of the wall to the base of the ladder using two different methods.
5. Explain how to find the length of the hypotenuse of $\triangle DEF$ without using the Pythagorean Theorem.

![Diagram of triangle DEF]

6. Jo says she can find $BC$ using the equation $\sin 40^\circ = \frac{BC}{62}$. Liam says he can find $BC$ using the equation $\cos 50^\circ = \frac{BC}{62}$. Who is correct? Explain.

![Diagram of triangle ABC]

**LESSON 22-3 PRACTICE**

7. Find each unknown side length.
   
   a. $x = \underline{\hspace{2cm}}$  
      $y = \underline{\hspace{2cm}}$
   
   b. $x = \underline{\hspace{2cm}}$  
      $y = \underline{\hspace{2cm}}$
   
   c. $x = \underline{\hspace{2cm}}$  
      $y = \underline{\hspace{2cm}}$

![Diagrams of triangles with side lengths]

8. **Reason quantitatively.** Jackson has the triangular garden shown in the figure. Find the perimeter and area of the garden. Be sure to check that your answers are reasonable.

![Diagram of a garden]

9. **Construct viable arguments.** The longer diagonal of a rhombus measures 20 cm. The rhombus has an angle that measures $100^\circ$. Determine the perimeter of the rhombus, to the nearest tenth. Explain how you found the answer.
Learning Targets:

• Calculate angle measures from trigonometric ratios.
• Solve right triangles.

SUGGESTED LEARNING STRATEGIES: Predict and Confirm, Summarizing, Quickwrite

1. In a right triangle $ABC$ with acute angle $A$, you know that $\sin A = \frac{\sqrt{3}}{2}$
   a. Draw a possible right triangle $ABC$.

   b. Determine what must be true about $\angle A$ using what you know about special right triangles.

In Item 1, you are given the sine of acute angle $A$ and are asked to find the angle whose sine is equal to that ratio. In other words, you are finding an inverse sine function. This is written as $\sin^{-1} \left( \frac{\sqrt{3}}{2} \right) = 60^\circ$. The expression $\sin^{-1} x$ is read as “the inverse sine of $x$.”

The **inverse trigonometric functions** for sine, cosine, and tangent are defined as follows:

<table>
<thead>
<tr>
<th>Inverse Trig Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $\sin A = x$, then $\sin^{-1} x = m\angle A$.</td>
</tr>
<tr>
<td>If $\cos A = x$, then $\cos^{-1} x = m\angle A$.</td>
</tr>
<tr>
<td>If $\tan A = x$, then $\tan^{-1} x = m\angle A$.</td>
</tr>
</tbody>
</table>

2. Use a scientific or graphing calculator to evaluate each of the following to the nearest tenth. Make sure your calculator is in DEGREE mode.

   a. $\sin^{-1} \left( \frac{1}{2} \right) =$

   b. $\cos^{-1} \left( \frac{1}{2} \right) =$

   c. $\tan^{-1} \left( \frac{1}{2} \right) =$

   d. $\sin^{-1} \left( \frac{3}{4} \right) =$

   e. $\cos^{-1} \left( \frac{3}{4} \right) =$

   f. $\tan^{-1} \left( \frac{3}{4} \right) =$

Using known measures to find all the remaining unknown measures of a right triangle is known as **solving a right triangle**.
Tricia designs the logo shown for one of her clients. She needs to know all the missing dimensions and angle measures of the logo.

3. **Model with mathematics.** Use an inverse trigonometric function to find the measure of angle $Q$.

4. Describe two ways to find the measure of angle $R$, without finding the length of the hypotenuse. Find the angle measure using both methods.

5. Describe two ways to find the length of the hypotenuse. Find the length using both methods.

**Check Your Understanding**

6. What is the minimal amount of information needed to solve a right triangle? Explain.

7. Explain the difference between the two expressions $\sin 15^\circ$ and $\sin^{-1}(0.2558)$. How are the expressions related?

**LESSON 22-4 PRACTICE**

8. Angle $X$ is an acute angle in a right triangle. What measure of angle $X$ makes each statement true? Round angle measures to the nearest tenth.
   a. $\cos X = 0.59$
   b. $\tan X = 3.73$
   c. $\sin X = 0.87$
   d. $\tan X = 0.18$
   e. $\cos X = 0.02$
   f. $\sin X = 0.95$

9. Solve each right triangle if possible. Round your measures to the nearest tenth.
   a. [Diagram of triangle with angle $49^\circ$ and side $27$]
   b. [Diagram of triangle with angle $18^\circ$ and side $15$
   c. [Diagram of triangle with angle $72^\circ$
   d. [Diagram of triangle with angle $68^\circ$

10. **Construct viable arguments.** Without using a calculator, explain how you can find the value of $\tan^{-1}(1)$.

11. Consider rhombus $ABCD$, where $AC = 8$ in. and $BD = 12$ in. What are the measures of the sides and angles of the rhombus? Round your answers to the nearest tenth.
**ACTIVITY 22 PRACTICE**

Write your answers on notebook paper. Show your work.

**Lesson 22-1**

1. Which side of $\triangle ABC$ is the side opposite angle $A$?

   A. $\overline{AB}$
   
   B. $\overline{AC}$
   
   C. $\overline{BC}$

2. In $\triangle PQR$, identify the hypotenuse, adjacent leg, and opposite leg for $\angle R$.

3. a. Find the missing measures in the given triangle.

   b. Draw and label a triangle similar to the triangle given in Part a. Include each side length and angle measure.

   c. State the scale factor of the triangle given in Part a to the triangle you drew in Part b.

4. Lisa wants to make a larger bandana similar to the bandana shown. If the shorter leg of the larger bandana is 15 inches, how long is its hypotenuse?

**Lesson 22-2**

5. Use your calculator to evaluate the following. Round to 3 decimal places.
   a. $\cos 54^\circ$
   
   b. $\sin 12^\circ$
   
   c. $\tan 67^\circ$

6. Find each of the following ratios. Write each ratio in simplest form.

   a. $\sin M$
   
   b. $\cos M$
   
   c. $\tan M$
   
   d. $\sin O$
   
   e. $\cos O$
   
   f. $\tan O$

7. Which expression is equivalent to $\cos 25^\circ$?
   A. $\cos 65^\circ$
   
   B. $\tan 65^\circ$
   
   C. $\sin 25^\circ$
   
   D. $\sin 65^\circ$

**Lesson 22-3**

8. Find the perimeter and area of each triangle.

   a.
   
   b.
   
   c.
9. Find \(x\) and \(y\). Round final answers to tenths.

\[
\begin{array}{c}
\text{y} \\
\text{18} \\
\text{42°} \\
\text{x} \\
\end{array}
\]

10. A badminton net is tethered to the ground with a strand of rope that forms a 50° angle with the ground. What is the length of the rope, \(x\)?

\[
\begin{array}{c}
\text{x} \\
\text{3.5 ft} \\
\text{50°} \\
\end{array}
\]

11. Use a scientific or graphing calculator to evaluate each of the following to the nearest tenth.

- a. \(\sin^{-1} \frac{1}{2} = \)
- b. \(\cos^{-1} \frac{1}{4} = \)
- c. \(\tan^{-1} 0.6 = \)

12. Solve each right triangle if possible. Round your measures to the nearest tenth.

- a.

\[
\begin{array}{c}
\text{B} \\
\text{8} \\
\text{62°} \\
\text{A} \\
\end{array}
\]

- b.

\[
\begin{array}{c}
\text{D} \\
\text{8.5} \\
\text{F} \\
\text{6.5} \\
\text{E} \\
\end{array}
\]

13. A skateboard ramp has a slope of \(\frac{4}{9}\). What is the measure of the angle the ramp forms with the ground?

14. Which expression is equivalent to \(\sin^{-1} (0.5)\)?

- A. \(\cos^{-1} (0.5)\)
- B. \(\cos^{-1} (0.5)\)
- C. \(90° - \cos^{-1} (0.5)\)
- D. \(90° - \tan^{-1} (0.5)\)

MATHEMATICAL PRACTICES

Look For and Make Use of Structure

15. Compare the values of the sine and cosine ratios as the measure of an angle increases from 0° to 90°.
The Law of Sines and Cosines

Learning Targets:
• Prove the Law of Sines.
• Apply the Law of Sines.

SUGGESTED LEARNING STRATEGIES: Create Representations, Graphic Organizer, Predict and Confirm, Visualization, Think-Pair-Share

The location of a fire spotted from two fire observation towers can be determined using the distance between the two towers and the angle measures from the towers to the fire. This process is known as triangulation.

You have already solved right triangles. In this lesson you will learn how to solve any triangle.

1. Begin with triangle ABC. Draw altitude CD from vertex C to AB. Label the altitude h.
2. What two right triangles are formed?
3. Reason abstractly. You can use the triangles you just formed to write some trigonometric ratios.
   a. Write a ratio for sin A.
   b. Write a ratio for sin B.

4. a. Solve each equation from Item 3 for h.
   b. Set the values for h equal.
   c. Complete the following statement.
      \[
      \frac{\sin A}{a} = \frac{\sin B}{b}
      \]
In a similar way, you can show \( \frac{\sin A}{a} = \frac{\sin C}{c} \) and \( \frac{\sin B}{b} = \frac{\sin C}{c} \).

You have just derived the **Law of Sines**.

Below is a formal statement of the Law of Sines.

For any triangle \( ABC \), with side lengths \( a, b, \) and \( c \),

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.
\]

**Example A**

Fire spotters at stations located at \( A \) and \( B \) notice a fire at location \( C \). What is the distance between station \( A \) and the fire?

\[
\begin{align*}
\angle &= \angle - \angle - \angle \\
&= \angle \\
&= 94^\circ
\end{align*}
\]

**Step 1:** Find \( m\angle C \).

\[
m\angle C = 180^\circ - 42^\circ - 44^\circ = 94^\circ
\]

**Step 2:** Use the Law of Sines.

\[
\frac{\sin B}{b} = \frac{\sin C}{c}
\]

\[
\begin{align*}
\sin 44^\circ &= \sin 94^\circ \\
\sin 44^\circ &= \frac{15 \sin 94^\circ}{b} \\
15 \sin 44^\circ &= b \cdot \sin 94^\circ \\
b &= \frac{15 \sin 44^\circ}{\sin 94^\circ} \\
b &\approx 10.4
\end{align*}
\]

**Solution:** The distance between station \( A \) and the fire is about 10.4 km.
Lesson 23-1
The Law of Sines

Try These A
Two boaters located at points A and B notice a lighthouse at location C. What is the distance between the boater located at point B and the lighthouse? Round to the nearest tenth.

Check Your Understanding

5. Reason quantitatively. Show that the Law of Sines is true for a $45^\circ$-$45^\circ$-$90^\circ$ triangle with leg lengths of 1.

LESSON 23-1 PRACTICE

7. Find each measure. Round to the nearest tenth.
   a. $YZ = \ldots$
   b. $DE = \ldots$

8. Construct viable arguments. When can you use the Law of Sines to find the measure of an angle of a triangle? Explain your thinking.

MATH TIP
You may need to use the Law of Sines to find an angle measure before finding the measure of a side. This may require the use of an inverse trigonometric function.
Learning Targets:
- Understand when the ambiguous case of the Law of Sines occurs.
- Solve problems using the Law of Sines.

**SUGGESTED LEARNING STRATEGIES:** Create Representations, Use Manipulatives, Predict and Confirm

Fire spotters at stations located at Q and R notice a fire at location S. The distance between stations Q and R is 15 km, and the distance between station R and the fire is 12 km. Station Q forms a 38° angle with station R and the fire. What is the distance between station Q and the fire?

1. Use appropriate tools strategically. Cut straws or coffee stirrers that are 15 cm and 12 cm long to model the distances between the locations. Use a protractor to form a 38° angle at vertex Q. How many different triangles can you make using a 38° angle and side lengths of 15 cm and 12 cm? Sketch the triangles.

2. Why could you form two different triangles in Item 1?

3. Is there an SSA criterion for proving two triangles are congruent? How does this help explain your answer to Item 2?

The problem you just explored is an example of the ambiguous case of the Law of Sines.

4. You know two side lengths and an angle measure. Use the Law of Sines to complete the equation to find the measure of angle S. Round to the nearest degree.
   a. \(\frac{\sin 38^\circ}{S} = \frac{\sin S}{S}\)
   b. \(m\angle S \approx \square^\circ\)

**TECHNOLOGY TIP**
You can also use geometry software to complete Item 1.
Lesson 23-2
The Ambiguous Case

5. There are two angles between 0° and 180° whose sine is 0.7660, one acute and one obtuse. Your calculator gives you the measure of the acute angle, 50°. The obtuse angle uses 50° as a reference angle.
   a. Find the measure of the obtuse angle whose sine is 0.7660.
      \[ m\angle S = 180° - 50° = \boxed{130°} \]
   b. Now find the measure of \( \angle R \) for the two possible triangles.
      \[ m\angle R = 180° - 38° - \boxed{50°} = \boxed{92°} \]
      \[ m\angle R = 180° - 38° - \boxed{50°} = \boxed{92°} \]
   c. Use the Law of Sines to complete the equations to find the two possible values of QS. Round to the nearest tenth.
      \[ \frac{\sin 38°}{12} = \frac{\sin \boxed{?}°}{QS} \]
      \[ QS = \boxed{?} \text{ or } QS = \boxed{?} \]

Check Your Understanding

6. Given: \( \triangle ABC \) with \( m\angle B = 40°, AB = 12, AC = 8 \). Find two possible values for each measure.
   a. \( m\angle C \)
   b. \( m\angle A \)
   c. \( BC \)

LESSON 23-2 PRACTICE

7. Given: \( \triangle ABC \) with \( m\angle A = 70°, BC = 85, AB = 88 \). Find two possible values for each measure.
   a. \( m\angle C \)
   b. \( AC \)

8. Given: \( \triangle ABC \) with \( m\angle A = 40°, BC = 26, AB = 32 \). Find two possible values for each measure.
   a. \( m\angle C \)
   b. \( AC \)

9. Reason abstractly. Three radio towers are positioned so that the angle formed at vertex A is 65°. The distance between tower A and tower B is 28 miles. The distance between tower B and tower C is 26 miles. What are the two possible distances between tower A and tower C? Draw a diagram to support your answers. Round to the nearest whole number.
Learning Targets:
• Prove the Law of Cosines.
• Solve problems using the Law of Cosines.

SUGGESTED LEARNING STRATEGIES: Create Representations, Predict and Confirm, Think-Pair-Share

Fire spotters at stations located at A and B notice a fire at location C. What is the distance between station B and the fire?

1. Can you use the Law of Sines to solve the problem? Explain.

You need a different relationship to solve this problem. You can solve the problem using the Law of Cosines.

For any triangle $ABC$, with side lengths $a$, $b$, and $c$,

\[
\begin{align*}
    a^2 &= b^2 + c^2 - 2bc \cos A \\
    b^2 &= a^2 + c^2 - 2ac \cos B \\
    c^2 &= a^2 + b^2 - 2ab \cos C
\end{align*}
\]

Follow these steps to prove the Law of Cosines.

2. Begin with triangle $ABC$. Draw altitude $\overline{CD}$ from vertex C to $\overline{AB}$.
   a. Label the altitude $h$.
   b. Label $\overline{AB}$ as $c$.
   c. Label $\overline{AD}$ in terms of $x$ and $c$.

3. Complete the following ratio.
   \[
   \cos A = \frac{x}{c}
   \]
Lesson 23-3
The Law of Cosines

4. Solve the equation from Item 3 for $x$.

5. Use the Pythagorean Theorem to complete the following statements.
   a. $x^2 + \underline{ } = b^2$
   b. $\underline{ }^2 + h^2 = a^2$

6. **Make sense of problems.** Solve for $h^2$ in each equation. Expand the equation from Item 5 and set them equal to each other. Then use substitution to prove the Law of Cosines.

7. Suppose $\angle B$ was a right angle. Show how the Law of Cosines would relate to the Pythagorean Theorem.

The Law of Cosines is a generalization of the Pythagorean Theorem.

Now you can solve the problem that was posed at the beginning of the lesson.

8. Fire spotters at stations located at $A$ and $B$ notice a fire at location $C$. What is the distance between station $B$ and the fire? Round to the nearest tenth.
   a. Use the Law of Cosines to complete the equation.
      $CB^2 = 10^2 + \underline{ }^2 - 2(\underline{ })\cos(\underline{ })$
   b. Solve the equation for $CB$. Round to the nearest tenth.
      The distance between station $B$ and the fire is _____.

**MATH TIP**

The **Pythagorean Theorem** states that the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the legs of the right triangle. In other words, for a right triangle with hypotenuse $c$ and legs $a$ and $b$, $c^2 = a^2 + b^2$. 
9. Fire station towers are located at points $A$, $B$, and $C$. The distances between the towers are known. Spotters need to know the measures of angles $A$, $B$, and $C$. What is the measure of angle $C$? Round to the nearest degree.

$$\triangle ABC$$

- $8 \text{ km}$
- $5 \text{ km}$
- $11 \text{ km}$

a. Which equation of the Law of Cosines do you use to find the measure of angle $C$?

b. Write the equation.

c. What is the measure of angle $C$?

**Check Your Understanding**

10. Given $\triangle XYZ$ and the lengths of $XY$ and $XZ$, do you have enough information to use the Law of Cosines to find $m\angle X$? Explain.

11. An equilateral triangle has sides of length 1. Use the Law of Cosines to show that each angle of the triangle measures $60^\circ$.

**LESSON 23-3 PRACTICE**

12. **Make use of structure.** Once you know the measure of angle $C$ in Item 9, describe two ways to find the measures of the remaining angles of the triangle.

13. Find each measure. Round to the nearest tenth.

a. $QR = \underline{\text{_____}}$  
   b. $m\angle D = \underline{\text{_____}}$
Learning Targets:

- Determine when to use the Law of Sines and when to use the Law of Cosines.
- Solve problems using the Law of Cosines and/or the Law of Sines.

SUGGESTED LEARNING STRATEGIES: Create Representations, Predict and Confirm, Think-Pair-Share, Visualization

To solve a triangle, you find all its angle measures and all its side lengths. It is important that you know when you can use the Law of Sines and when you can use the Law of Cosines to solve triangles.

1. Complete the table below and use it as a reference.

<table>
<thead>
<tr>
<th>Case</th>
<th>Description</th>
<th>Use Law of Sines or Law of Cosines to solve?</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAS</td>
<td>Two angle measures and the length of a nonincluded side are known.</td>
<td></td>
</tr>
<tr>
<td>ASA</td>
<td>Two angle measures and the length of the included side are known.</td>
<td></td>
</tr>
<tr>
<td>SSA</td>
<td>The lengths of two sides and the angle opposite are known.</td>
<td></td>
</tr>
<tr>
<td>SAS</td>
<td>Two side lengths and the included angle measure are known.</td>
<td></td>
</tr>
<tr>
<td>SSS</td>
<td>Three side lengths are known.</td>
<td></td>
</tr>
</tbody>
</table>

2. Fire spotters at stations located at B and C notice a fire at location A. Solve the triangle.
   a. What information do you know?

   ![Triangle Diagram]

   - A: 98°, 15 km, 12 km
   - B: 12 km
   - C: 98°, 15 km

   b. Describe the first step in solving the problem.

   c. Make sense of problems. Draw a flowchart to plan a solution pathway for the problem.
d. Does your plan include using the Law of Sines, the Law of Cosines, or both? Explain.

e. Solve the triangle. Round all side measures to the nearest tenth and angle measures to the nearest degree.

Check Your Understanding

3. Suppose you know the lengths of the three sides of a triangle. Describe how you would find the measure of each angle of the triangle.

4. Suppose you know the measures of the three angles of a triangle. Can you use the Law of Sines and/or the Law of Cosines to solve the triangle? Explain.

LESSON 23-4 PRACTICE

5. Solve each triangle. Round all side measures to the nearest tenth and angle measures to the nearest degree.

6. Reason quantitatively. Two lifeguards on the towers at vertex A and vertex B are watching a swimmer in the ocean at vertex C. Which lifeguard is closer to the swimmer? Show your work.
ACTIVITY 23 PRACTICE
Write your answers on notebook paper. Show your work.

Unless otherwise indicated, round all side measures to the nearest tenth and angle measures to the nearest degree.

Lesson 23-1

1. Solve for \( x \):
   \[ \sin 28^\circ = \sin 52^\circ \]
   \[ x \]

2. Solve for \( A \):
   \[ \frac{\sin 82^\circ}{28} = \frac{\sin A^\circ}{8} \]

3. Describe how to use the Law of Sines to find the lengths of the other two sides of a triangle, if you know the measures of two angles and the included side.

4. To the nearest tenth, what is \( RS \)?

   \[ R \]
   \[ 28 \]
   \[ Q \]
   \[ 49^\circ \]
   \[ 37^\circ \]
   \[ S \]

   A. 21.2  
   B. 29  
   C. 35.1  
   D. 37

5. Find each measure.
   a. \( AC = \____\)

   \[ C \]
   \[ 85^\circ \]
   \[ 9 \text{ km} \]
   \[ A \]
   \[ 48^\circ \]
   \[ B \]

   b. \( HJ = \____\)

   \[ H \]
   \[ 8 \text{ km} \]
   \[ G \]
   \[ 61^\circ \]
   \[ 24^\circ \]
   \[ J \]

   c. \( m\angle R = \____\)

   \[ S \]
   \[ 42^\circ \]
   \[ 36 \]
   \[ R \]
   \[ 70^\circ \]
   \[ T \]

Lesson 23-2

6. Given: \( \triangle ABC \) with \( m\angle A = 28^\circ \), \( BC = 6 \), \( AB = 11 \).
   Find two possible values for each measure.
   a. \( m\angle C \)
   b. \( AC \)

7. Given: \( \triangle ABC \) with \( m\angle A = 65^\circ \), \( BC = 19 \), \( AB = 21 \).
   Find two possible values for each measure.
   a. \( m\angle C \)
   b. \( AC \)

8. Three fire towers are positioned so that the angle formed at vertex \( R \) is \( 31^\circ \). The distance between tower \( R \) and tower \( S \) is 8 miles. The distance between tower \( S \) and tower \( T \) is 6 miles. What are the two possible distances between tower \( R \) and tower \( T \)?

Lesson 23-3

9. Solve for \( b \):
   \[ b^2 = 22^2 + 28^2 - 2(22)(28)\cos 36^\circ \]

10. Solve for \( C \):
    \[ 5^2 = 8^2 + 11^2 - 2(8)(11)\cos C^\circ \]
11. To the nearest degree, what is $m\angle S$?

A. 34°  
B. 40°  
C. 50°  
D. 106°

12. Find each measure.
   a. $m\angle C =$ _____

   b. $m\angle S =$ _____

13. What is the distance across the lake, $AB$?

14. Solve each triangle.
   a. 

   b. 

15. Chandra wants to find the measure of angle $B$ in triangle $ABC$. Is her work correct? Explain. If it is not, then fix the error and find the measure of angle $B$.

   $$\cos B = \frac{36^2 - 20^2 + 30^2}{2(20)(30)}$$
Chloe wants to ride a zip line. She looked online and found a zip line tour near her home, and she needs to decide whether to ride the Beginner’s zip line or the Daredevil’s zip line. The tour company’s Web page gives some of the measurements for the two zip lines, but Chloe wants to find some additional measurements before she decides. The image below is shown on the tour company’s Web page.

1. Chloe printed the image and added labels to help her identify sides and angles. Describe how to find \( BC \), the difference in the heights of the starting points.

2. Find each of the following measures. Round your answer to the nearest whole number or degree.
   a. \( BC = \) ______
   b. \( m\angle B = \) ______
   c. \( m\angle BCA = \) ______
   d. \( m\angle CAD = \) ______
   e. \( m\angle DCA = \) ______
   f. \( AD = \) ______

3. What is the height of the starting point of the Beginner’s zip line, \( CD \)?
4. What is the height of the starting point of the Daredevil’s zip line, \( BD \)?

To prepare for her zip line adventure, Chloe’s father helps her build a smaller zip line in her backyard. Chloe made a diagram of her zip line.

5. The height of the starting point, \( SB \), is 20 feet and \( EB = 26 \) feet. Determine the angle of elevation from the ending platform, \( E \), to the starting point, \( S \). Show the work that leads to your response. Round your angle to the nearest degree.
6. Chloe decides she wants the length of the wire for the zip line to be 60 feet long and the angle of elevation to be $40^\circ$. Chloe and her father need to determine the distance from $B$ to $E$ to decide if there is enough room in her yard for this zip line. Her father uses the equation 

$$\sin 50^\circ = \frac{EB}{60}$$

and Chloe uses the equation 

$$\cos 40^\circ = \frac{EB}{60}.$$ 

Who is correct? Justify your answer.